Advanced and algorithmic graph theory

Summer term 2018

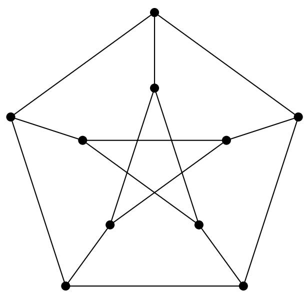
Exercise sheet 5

Exercises for the exercise session on 07/06/2018

Problem 5.1. Along the lines of Brooks' theorem, derive an algorithm that finds, for every connected graph G that is neither complete nor an odd cycle, a $\Delta(G)$-colouring in time $O(m+n)$.

Problem 5.2. Prove that the recursive-largest-first algorithm colours all bipartite graphs optimally and show that it can be implemented to run in time $O(nm)$.

Problem 5.3. Let $n \geq 2$ be an integer. Prove that $\chi'(K^n) = n$ if n is odd and $\chi'(K^n) = n-1$ if n is even. Furthermore, determine the chromatic index of the Petersen graph depicted below.



Problem 5.4. Prove directly (that is, without using any results about edge-colourings from the lecture) that every k-regular bipartite graph is k-edge-colourable. Prove that this implies Theorem 4.22, i.e. $\chi'(G) = \Delta(G)$ for every bipartite graph.

Problem 5.5. For every $k \in \mathbb{N}$, construct a bipartite graph G_k and an assignment of lists that shows that G_k is *not* k-choosable.

Bonus problem. For a positive integer r, we denote by K_2^r the graph with vertex set the disjoint union of sets $V_1, \ldots, V_r$ of size 2 and, for all $1 \leq i < j \leq r$, all edges between V_i and V_j. Prove that $\mathrm{ch}(K_2^r) = r$.

Hint. Try to use induction on r. If the induction step fails, what does this tell us about the lists?

Problem 5.6. A *total colouring* of G is a function $c\colon V(G) \cup E(G) \to S$ such that $c|_{V(G)}$ and $c|_{E(G)}$ are vertex- and edge-colourings, respectively, and in addition no edge has the same colour as one of its end vertices. We write $\chi_t(G)$ for the least k for which there exists a total colouring of G with k colours.

Prove that the list colouring conjecture would imply $\Delta(G)+1 \leq \chi_t(G) \leq \Delta(G)+3$. (The *total colouring conjecture* asserts that even $\chi_t(G) \leq \Delta(G)+2$.)