Advanced and algorithmic graph theory

Summer term 2018

Exercise sheet 6

Exercises for the exercise session on 28/06/2018

Problem 6.1. Pick one of the proofs of Dirac's theorem and show that it can also be applied to prove Ore's theorem (possibly with some minor changes). Based on the proof, describe an algorithm that finds a Hamilton cycle in a given graph G that satisfies the condition of Ore's theorem. What running time can you achieve? What running time is necessary to check whether G satisfies the condition of Dirac's or Ore's theorem, respectively?

Problem 6.2. Suppose that G has at least two vertices and that $d(x) + d(y) \geq |G|$ for all distinct non-adjacent $x, y \in V(G)$. Prove that $\kappa(G) \geq \alpha(G)$.

(In particular, the Chvátal-Erdős theorem implies Ore's theorem.)

Hint. Given a separator S, vertices with more than $|S|$ common neighbours will lie in the same component of $G - S$.

Problem 6.3. Recall that the square G^2 of a graph G has the same vertex set as G and an edge between v and w iff v and w have distance at most two in G.

(a) Prove that if G is k-connected (for some positive integer k), then G^2 is k-tough.

(b) Find a connected graph G on at least three vertices for which G^2 is not hamiltonian.

Note. This proves that not every 1-tough graph is hamiltonian.

Problem 6.4. Let T be a tree with $\|T\| \geq 2$. The *Erdős-Sós Conjecture* states that

$$\mathrm{ex}(n, T) \leq \frac{1}{2}(\|T\| - 1)n.$$

(a) Show that the Erdős-Sós Conjecture is true if $T = K_{1,r}$ (with $r \geq 2$).

(b) Prove the weaker statement

$$\mathrm{ex}(n, T) \leq \|T\|\, n.$$

Hint. Large minimum degree would make it easy to find T as a subgraph.

(c) Show that the Erdős-Sós Conjecture is best possible in the sense that

$$\mathrm{ex}(n, T) \geq \frac{1}{2}(\|T\| - 1)n$$

for infinitely many values of n. (These values might depend on T.)

Problem 6.5. Let k, r be positive integers.

(a) Given a graph G with $d(G) \geq 2k$, let H be the smallest induced subgraph of G which satisfies both $\delta(H) \geq k$ and $\|H\| \geq k(|H| - k)$. (Why does such an H exist?) Prove that H is k-connected.

(b) Show that every $8r^2$-connected graph contains a TK^r. Deduce that this implies Theorem 6.7 from the lecture with $c = 16$.

Hint. Apply Theorem 6.8 to link a suitable choice of vertices.

Problem 6.6. Let $\varepsilon > 0$ and integers $r_1 > r_2 \geq 1$ be given.

(a) Prove that Hadwiger's Conjecture for $r = r_1$ implies the conjecture for $r = r_2$.

(b) Show that there exists r_0 such that for each integer $r \geq r_0$, every graph G with $\chi(G) \geq r^{1+\varepsilon}$ contains an MK^r.