# Analytic combinatorics

SS 2018

## Exercise sheet 1

Exercises for the exercise session on 12 March 2018

**Problem 1.1.** Let $\mathcal{T}$ be the class of $r$-nary trees and $\mathcal{T}_n$ the class of $r$-nary trees of size $n$.

- Express $\mathcal{T}$ in terms of $\mathcal{T}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.
- Derive a closed formula for $\mathcal{T}_n$ and its asymptotic formula.

**Problem 1.2.** Let $\mathcal{U}$ be the class of unary-binary trees and $\mathcal{U}_n$ the class of unary-binary trees of size $n$.

- Express $\mathcal{U}$ in terms of $\mathcal{U}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.
- Derive a closed formula for $\mathcal{U}_n$.

**Problem 1.3.** This question concerns the number of ways a string of $n$ identical letters, say $x$, can be 'bracketed'. The rule is best stated recursively: $x$ itself is a bracketing and if $\sigma_1, \sigma_2, \ldots, \sigma_k$ with $k \geq 2$ are bracketed expressions, then the $k$-ary product $(\sigma_1\sigma_2\cdots\sigma_k)$ is a bracketing. For instance: $(((xx)x(xxx))((xx)(xx)x))$. Let $\mathcal{S}$ denote the class of all bracketings, where size is taken to be the number of instances of $x$.

- Derive a recursive formula for $s_n := |\mathcal{S}_n|$ and derive the corresponding expression in terms of generating functions.
- Derive a closed formula for $s_n$ and its asymptotic formula.

**Problem 1.4.** Consider a sequence of numbers $x = (x_0 = 0, x_1, \ldots, x_{2n-1}, x_{2n} = 0)$ satisfying $x_i \geq 0, \quad |x_i - x_{i-1}| = 1$ for $1 \leq i \leq 2n$. This represents an excursion that take place in the upper half-plane, also known as Dyck paths of length $2n$. Let $\mathcal{D}$ be the class of Dyck paths and $\mathcal{D}_{2n}$ the class of Dyck paths of length $2n$.

- Express $\mathcal{D}$ in terms of $\mathcal{D}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.
- Derive a closed formula for $\mathcal{D}_{2n}$ and its asymptotic formula.

**Problem 1.5.** A meander is a word over $\{-1, +1\}$ such that the sum of the values of any of its prefixes is a non-negative integer. A bridge is a word over $\{-1, +1\}$ whose values of its letters sum to 0. Note that a meander represents a walk that wanders in the first quadrant, and a brige a walk that wanders above and below the horizontal line, but its final altitute is constrained to be 0. Let $\mathcal{M}$ be the class of meanders and $\mathcal{B}$ the class of bridges.

- Express $\mathcal{M}$ and $\mathcal{B}$ in terms of $\mathcal{D}$ and basic constructions and derive the corresponding expression in terms of generating functions.
- Derive a closed formula for $\mathcal{M}_{2n}$ and $\mathcal{B}_{2n}$.