Analytic combinatorics
SS 2018

Exercise sheet 2

Exercises for the exercise session on 21 March 2018

Problem 2.1. Recall that the class of mappings from $\{1, 2, \dots, n\}$ to itself has exponential generating function

$$F(z) = \frac{1}{1 - T(z)},$$

where $T(z) = ze^{T(z)}$. Drive a closed formula for

$$[z^n]F(z)$$

(hint: you may use the Lagrange formula $[z^n]G(y(z)) = \frac{1}{n}[u^{n-1}]G'(u) \cdot (\phi(u))^n$ where $y(z)$ satisfy $y = z\phi(y)$).

Problem 2.2. Recall that the class of mappings from $\{1, 2, \dots, n\}$ to itself with no fixed points has exponential generating function

$$H(z) = \frac{e^{-T(z)}}{1 - T(z)},$$

where $T(z) = ze^{T(z)}$. Drive a closed formula for

$$[z^n]H(z).$$

Problem 2.3. Recall that the exponential bivariate generating function of permutations counted according to cycles is $p(z, u) = (1 - z)^{-u}$. Let χ denote the number of cycles in a random permutation, which is chosen uniformly at random among all permutations of size n. Derive a closed formula for the second factorial moment of χ

$$\mathbb{E}(\chi(\chi - 1))$$

(you may use the binomial theorem for negative exponents $(1 - z)^{-u} = \sum_{n \geq 0} \binom{u+n-1}{n} z^n$).

Problem 2.4. Let C_n denote a random Cayley tree, which is chosen uniformly at random from the set of all Cayley trees on n vertices and P_n a random plane tree, which is chosen uniformly at random from the set of all plane trees on n vertices.

(a) Determine the average degree of the root in C_n

(b) Determine the average degree of the root in P_n.