Analytic combinatorics
SS 2018

Exercise sheet 3

Exercises for the exercise session on 16 April 2018

Problem 3.1. Consider a sequence $\{D_n\}_{n\geq 1}$ satisfying the recurrence

$$\begin{cases} D_n = D_{\lfloor \frac{n}{2} \rfloor} + 1, & n \geq 2, \\ D_1 = 2. \end{cases}$$

Determine closed formulae for

(1) D_n

(2) $\sum_{k=1}^{n} D_k$.

Problem 3.2. Find initial conditions a_0, a_1 and a_2 for which the growth rate of the solution to the recurrence

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \quad n \geq 3$$

is

(1) constant

(2) exponential and

(3) fluctuating in sign.

Problem 3.3. (1) Prove the identity $\sum_{0\leq k\leq n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$.

(2) What identity on binomial coefficient is implied by the convolution

$$(1+z)^r(1-z)^s = (1-z^2)^s(1+z)^{r-s}$$

for $r > s$?

(3) Prove the identity $\sum_{0\leq k\leq t} \binom{t-k}{r}\binom{k}{s} = \binom{t+1}{r+s+1}$.

Problem 3.4. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the harmonic number.

(1) Find the explicit OGF of the sequence $(H_n)_{n\geq 1}$ of harmonic numbers

$$\sum_{n\geq 1} H_n z^n = ?$$

(2) Prove the identity

$$[z^n]\left(\frac{z}{(1-z)^2} \ln \frac{1}{1-z}\right) = n(H_n - 1).$$

Problem 3.5. (1) Using EGF, find a solution to the recurrence

$$\begin{cases} a_n = \sum_{0\leq k\leq n} \binom{n}{k} \frac{a_k}{2^k}, & n \geq 1, \\ a_0 = 1. \end{cases}$$

(2) Consider an EGF $e^{z+\frac{z^2}{2}}$ of a sequence $\{a_n\}_{n\geq 0}$. Show that a_n satisfies the recurrence

$$a_n = a_{n-1} + (n-1)a_{n-2}, \quad n \geq 2.$$