Analytic combinatorics
SS 2018

Exercise sheet 5

Exercises for the exercise session on 28 May 2018

Problem 5.1. Let $C(T)$ denote the center of a tree $T = (V, E)$.

(1) Prove that $|C(T)| \leq 2$.

(2) Prove that if $C(T) = \{v, w\}$, then $\{v, w\} \in E$.

Problem 5.2. Let $G(z)$ denote the ordinary generating function for the class of unlabelled ternary trees. Using a general result on the tree enumeration with generating function $T(z) = z\phi(T(z))$,

(1) determine the dominant singularity ρ of $G(z)$;

(2) derive a singular expansion of $G(z)$ near ρ;

(3) find an asymptotic expression for $[z^n]G(z)$.

Problem 5.3. Let $M(z)$ denote the ordinary generating function for the class of unlabelled plane rooted trees in which each vertex has at most two descendants. Using a general result on the tree enumeration with generating function $T(z) = z\phi(T(z))$,

(1) determine the dominant singularity ρ of $M(z)$;

(2) derive a singular expansion of $M(z)$ near ρ;

(3) find an asymptotic expression for $[z^n]M(z)$.

Problem 5.4. Let $C(z)$ denote the exponential generating function for the class of labelled rooted trees (i.e. Cayley trees).

(1) Find an asymptotic expression for $[z^n]C(z)$, using a general result on the tree enumeration with generating function $T(z) = z\phi(T(z))$.

(2) Use Lagrange's Inversion Theorem to determine an asymptotic expression for $n![z^n]C(z)$, and use this to prove Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Problem 5.5. Let $T^\circ(z)$ denote the ordinary generating function for unlabelled rooted trees and let η be its radius of convergence. Let $C(z)$ be as in Problem 5.4 and define $R(z) = z\exp\left(\sum_{k\geq 2}\frac{1}{k}T^\circ(z^k)\right)$.

(1) Show that $T^\circ(z) = C(R(z))$ and η is equal to the smallest positive real solution of the equation $R(\eta) = 1/e$.

(2) Derive the following singular expansion of $T^\circ(z)$ near η:

$$T^\circ(z) = 1 + C_1(1 - z/\eta)^{1/2} + C_2(1 - z/\eta) + O\left((1 - z/\eta)^{3/2}\right).$$

(3) Determine the constants C_1, C_2 explicitly in terms of η and $R'(\eta)$.