Analytic combinatorics
SS 2018

Exercise sheet 6

Exercises for the exercise session on 6 June 2018

Problem 6.1. Let $T^\circ(z,w)$ denote the bivariate generating function for unlabelled rooted trees, where w marks the number of pendant copies of (a fixed unlabelled rooted tree) H. For each $w \in (1-\delta, 1]$ (for a given $0 < \delta < 1$) we let $\eta(w)$ denote the dominant singularity of $T^\circ(z,w)$. Prove that $\eta(w) \geq \eta(1)$.

Problem 6.2. Let $\mathcal{C}_n$ denote the class of labelled rooted trees (i.e. Cayley trees) on n vertices and let R_n be chosen uniformly at random from $\mathcal{C}_n$. Let H be a fixed rooted tree and let X_n denote the number of pendant copies of H in R_n. Prove that there exists a constant $\delta > 0$ such that

$$\mathbb{P}[X_n \leq \delta n] \;\leq\; e^{-\Omega(n)}.$$

Problem 6.3. Let S_n denote the number of ways to partition a labelled set of size n (so, for example, $S_3 = 5$ by considering the partitions $\{1\}\{2\}\{3\}$, $\{1\}\{2,3\}$, $\{2\}\{1,3\}$, $\{3\}\{1,2\}$, and $\{1,2,3\}$), and let $S(z) = \sum_{n\geq 0} \frac{S_n}{n!} z^n$ denote the corresponding exponential generating function.
Show

$$[z^n]S(z) \leq \frac{e^{n-1}}{(\ln n)^n}.$$

Problem 6.4.
(1) Let R_n be as in Problem 6.2 and let X_n denote the root degree of R_n. Determine $\lim_{n\to\infty} \mathbb{P}[X_n = k]$ for each $k \geq 1$.
(2) Let P_n be a uniform random unlabelled plane rooted tree and let Y_n denote the root degree of P_n. Determine $\lim_{n\to\infty} \mathbb{P}[Y_n = k]$ for each $k \geq 1$.

Problem 6.5. Show that the distribution of the number of cycles of length $\ell \in \mathbb{N}$ in a random permutation of size n converges to a Poisson distribution of rate $1/\ell$.