Probabilistic method in combinatorics and algorithmics
WS 2017/18

Exercise sheet 1

Exercises for the exercise session on 10/10/2017

Problem 1.1. Let G be a bipartite graph with n vertices and suppose that each vertex v has a list $S(v)$ of 'colours'. Prove that if $|S(v)| > \log_2 n$ for each v, then we can colour every vertex with a colour from its list so that no two adjacent vertices have the same colour.
Hint. Partition the set $\bigcup_v S(v)$ into two random sets.

Problem 1.2. Let $k = k(n)$ be such that $k \to \infty$ as $n \to \infty$, but $k = o(n^{2/3})$. Use Stirling's formula

$$n! = \big(1 + o(1)\big)\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n$$

and the Taylor expansion for $\ln(1+x)$ to prove that

$$\binom{n}{k} = \big(1 + o(1)\big)\frac{1}{\sqrt{2\pi k}}\left(\frac{n}{k}\right)^k \cdot \begin{cases} \exp(k) & \text{if } k = o(n^{1/2}), \\ \exp\left(k - \frac{k^2}{2n}\right) & \text{if } k = \Omega(n^{1/2}). \end{cases}$$

What exponential term would be needed if $k = \Omega(n^{2/3})$, but $k = o(n^{3/4})$?

Problem 1.3. Suppose an unbiased coin is tossed n times. For $k \le n$, let A_k denote the event that out of these n tosses, there are k consecutive ones with the same outcome (i.e. k consecutive 'heads' or k consecutive 'tails'). Let $\varepsilon > 0$. Prove that

(a) $\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} 0$ if $k \ge (1+\varepsilon)\log_2 n$;

(b) $\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} 1$ if $k \le \log_2 n - (1+\varepsilon)\log_2\log_2 n$.

Problem 1.4. A *claw* in a graph G are four vertices u, v, w, x in G such that G has no edges between u, v, w, but all three edges from x to u, v, and w. Denote by X the number of claws in $G(n,p)$. What is the expectation of X? For which p is this expectation of size $\Theta(1)$? For which p does the first moment method show that $\mathbb{P}(G(n,p)$ contains a claw$) \to 0$?

Problem 1.5. Let $k \in \mathbb{N}$ such that $p = 3k+2$ is a prime number and let $A \subset \mathbb{Z}_p \backslash \{0\}$. Prove that there exists some $b \in \mathbb{Z}_p$ with

$$\Big|A \cap \big\{b \cdot z \mid z \in \{k+1, \dots, 2k+1\}\big\}\Big| > \frac{|A|}{3}.$$

(The product $b \cdot z$ is of course to be taken in $\mathbb{Z}_p$.)
Deduce from this that there exists a $B \subset A$ with $|B| > \frac{1}{3}|A|$ such that for all $x, y, z \in B$, we have $x + y \neq z$. (Such a set is called *sum-free*.)