Probabilistic method in combinatorics and algorithmics

WS 2017/18

Exercise sheet 2

Exercises for the exercise session on 25/10/2017

Problem 2.1. Let $n \geq k \geq 1$ be integers.

(a) Prove that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{en}{k}\right)^k.$$

(b) For any constant $\alpha \in (0,1)$, show

$$\binom{n}{\alpha n} = 2^{H(\alpha)n + O(\log_2 n)},$$

where $H\colon (0,1) \to \mathbb{R}$ is defined by

$$H(x) = -x \log_2 x - (1-x)\log_2(1-x).$$

Prove that the same formula is still true if α is not constant, but satisfies

$$\alpha = \omega\left(\frac{1}{n}\right) \qquad \text{and} \qquad 1-\alpha = \omega\left(\frac{1}{n}\right).$$

Problem 2.2. Let $x \in \mathbb{R}$ and integers $n \geq k \geq 1$ be given.

(a) Prove that $1 + x \leq \exp(x)$. Furthermore, prove that $1 + x \geq \exp\left(x - \frac{x^2}{2}\right)$ is true if and only if $x \geq 0$.

(b) Use (a) to show that the falling factorial $(n)_k := \frac{n!}{(n-k)!}$ satisfies

$$n^k \exp\left(-\frac{k(k-1)}{2(n-k+1)}\right) \leq (n)_k \leq n^k \exp\left(-\frac{k(k-1)}{2n}\right).$$

Problem 2.3. Let $c \geq r \geq 2$ be integers. An r*-uniform hypergraph* H consists of a set V of vertices and a set $E \subseteq \binom{V}{r}$ of edges (i.e. a 2-uniform hypergraph is just a graph).

(a) Prove that the vertices of an r-uniform hypergraph with $|E| \leq c^r$ can be coloured with $2c-1$ colours so that no edge is monochromatic.

(b) Prove that the vertices of an r-uniform hypergraph with $|E| \leq \frac{2(c-r+1)^2}{r(r-1)^2}$ can be coloured with $2c-r$ colours so that every edge is 'rainbow coloured', that is, all its vertices have different colours.

Hint. In both cases, prove first that there is a colouring with c colours for which the desired property only fails for 'few' edges.

Problem 2.4. For $n, m \in \mathbb{N}$, we define the *uniform random graph* $G(n,m)$ to be a graph chosen uniformly at random from all graphs with vertex set $[n] := \{1, \ldots, n\}$ and m edges. Prove that for every $\varepsilon > 0$, we have

$$\mathbb{P}(G(n,m) \text{ contains an isolated vertex}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } m \geq \left(\frac{1}{2} + \varepsilon\right) n \ln n, \\ 1 & \text{if } m \leq \left(\frac{1}{2} - \varepsilon\right) n \ln n. \end{cases}$$

Problem 2.5. Call an edge in a graph *isolated* if both its end vertices lie in no other edge. Denote by X the number of isolated edges in $G(n,p)$.

(a) Determine $\mathbb{E}[X]$ and prove that for every given $\varepsilon > 0$,

$$\mathbb{E}[X] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p \geq n^{\varepsilon - 1}, \\ \infty & \text{if } p \leq (1-\varepsilon)\frac{\ln n}{2n}, \text{ but } p = \omega\left(\frac{1}{n^2}\right). \end{cases}$$

Hint. For $\mathbb{E}[X] \to \infty$, it might help to split the interval for p into two parts.

(b) Prove that

$$\mathbb{P}(X) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } \mathbb{E}[X] \to 0, \\ 1 & \text{if } \mathbb{E}[X] \to \infty. \end{cases}$$