Probabilistic method in combinatorics and algorithmics

WS 2017/18

Exercise sheet 3

Exercises for the exercise session on 15/11/2017

Problem 3.1. Let H be a fixed graph with k vertices and $m \geq 1$ edges. We define the *maximum density* m_H of H by

$$m_H := \max\left\{ \frac{|E(H')|}{|V(H')|} \;\middle|\; H' \text{ is a non-empty subgraph of } H \right\}.$$

Prove that

$$\mathbb{P}[H \text{ is a subgraph of } G(n,p)] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = o\left(n^{-\frac{1}{m_H}}\right), \\ 1 & \text{if } p = \omega\left(n^{-\frac{1}{m_H}}\right). \end{cases}$$

Note. For a given set $S \in \binom{[n]}{k}$, there might be more than one bijection $V(H) \to S$ that preserves adjacencies. At some point in your proof, you should show that the number of such bijections does not influence your arguments.

Problem 3.2. Let $r \geq 2$ be given. For any $n \in \mathbb{N}$, $p \in [0,1]$, we denote by $H_r(n,p)$ the random r-uniform hypergraph on n vertices, in which each element of $\binom{[n]}{r}$ forms an edge with probability p independently. Let A be the event that each $(r-1)$-tuple is contained in at least one edge. For fixed $\varepsilon > 0$, prove that

$$\mathbb{P}[A] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = (1-\varepsilon)\frac{(r-1)\ln n}{n}, \\ 1 & \text{if } p = (1+\varepsilon)\frac{(r-1)\ln n}{n}. \end{cases}$$

Suppose we replace ε by a function $\varepsilon(n) = o(1)$. How fast can $\varepsilon(n)$ tend to 0 if we still want to be able to prove the same result?

Problem 3.3. Let A_1, A_2, A_3 be events in a probability space such that the following holds.

- A_1 and A_2 are independent;
- A_2 and A_3 are independent;
- $A_1 \cap A_2$ and A_3 are independent.

Prove that A_1 and $A_2 \cap A_3$ are independent as well. Deduce from this that one may assume in the Lovász Local Lemma that the dependency graph D has the following properties.

(i) Every vertex i of D is either isolated (i.e. there are no edges involving i) or has at least one outgoing edge (i.e. an edge (i,j)).

(ii) Either D is edgeless or contains a directed cycle (this cycle might consist of only two edges (i,j) and (j,i)).

Problem 3.4. Let $n \geq 2$ be fixed. For the following probability spaces and sets of 'bad' events $A_1, \dots, A_n$, prove that the Lovász Local Lemma *cannot* be applied (regardless of the choice of the dependency graph D).

(a) Let Ω be the set of all sequences $s_1, \dots, s_n$ with $s_i \in \{1, \dots, 6\}$ such that $s_1 + \cdots + s_n$ is even. Choose an element from Ω uniformly at random. (I.e. we are considering the outcome of throwing a fair die n times, conditioned on the event that the sum of values is even.) Denote by A_i the event that s_i is odd.

(b) Let Ω be the set of all permutations of $[n]$. We choose a permutation σ uniformly at random and denote, for each $i \in [n]$, by A_i the event that $\sigma(i) = i$.

Problem 3.5. Let $A_1, \dots, A_n$ be events in a probability space and let $D = ([n], E)$ be a directed graph. We say that D is a *negative dependency graph* for $A_1, \dots, A_n$ if the following holds.

> For every $J \subseteq [n]$ and $i \in [n] \setminus J$, if $(i, j) \notin E$ for all $j \in J$ and $\mathbb{P}\left[\bigwedge_{j \in J} \overline{A_j}\right] > 0$, then $\mathbb{P}\left[A_i \,\middle|\, \bigwedge_{j \in J} \overline{A_j}\right] \leq \mathbb{P}(A_i)$.

Replacing 'dependency graph' by 'negative dependency graph' in the statement of the Lovász Local Lemma results in the so-called *Lopsided Lovász Local Lemma.*

(a) Sketch how the proof of the Lovász Local Lemma from the lecture generalises to a proof of the Lopsided Lovász Local Lemma.

(b) Follow the arguments sketched below to prove in that Problem 3.4(b), the edgeless graph on $[n]$ is a negative dependency graph.

It suffices to show (why?) that

$$\mathbb{P}\left[\bigwedge_{j \in J} \overline{A_j} \,\middle|\, A_i\right] \leq \mathbb{P}\left[\bigwedge_{j \in J} \overline{A_j}\right]$$

for all $J \subset [n]$ and $i \in [n] \setminus J$ such that $\mathbb{P}[A_i] > 0$. Show that this is equivalent to

$$n \left| A_i \wedge \bigwedge_{j \in J} \overline{A_j} \right| \leq \left| \bigwedge_{j \in J} \overline{A_j} \right|.$$

To prove this inequality, for each permutation $\sigma \in A_i \wedge \bigwedge_{j \in J} \overline{A_j}$, define n permutations $\sigma_1, \dots, \sigma_n \in \bigwedge_{j \in J} \overline{A_j}$ and prove that $\sigma_k \neq \tau_l$ as soon as $k \neq l$ or $\sigma \neq \tau$.