Probabilistic method in combinatorics and algorithmics
WS 2017/18

Exercise sheet 4

Exercises for the exercise session on 28/11/2017

Problem 4.1. Let $A_1, \dots, A_n$ be events in a probability space and let $D = ([n], E)$ be a dependency graph for the events $A_1, \dots, A_n$. Use the General Lovász Local Lemma to prove that the intersection of the events $\overline{A_1}, \dots, \overline{A_n}$ has a positive probability if one of the following holds.

(a) For every $i = 1, \dots, n$,

$$\mathbb{P}[A_i] < 1 \qquad \text{and} \qquad \sum_{(i,j)\in E} \mathbb{P}[A_j] \leq \frac{1}{4}.$$

Hint. Try $x_i = c\mathbb{P}[A_i]$ for a global constant c. At some point in the proof, you might want to use the (easy to show) inequality $(1-a)(1-b) \geq 1-(a+b)$ for $0 \leq a, b < 1$.

(b) (Symmetric LLL) There exist $d \geq 1$ and $p \in (0,1)$ such that

- no vertex in D has more than d outgoing edges,
- $\mathbb{P}[A_i] \leq p$ for all $i = 1, \dots, n$, and
- $ep(d+1) \leq 1$.

Problem 4.2. Let $k \geq 2$ be given.

(a) Suppose that H is a hypergraph in which each edge has at least k elements. For each edge f and each $j \geq k$, denote by $d_{f,j}$ the number of edges of size j that intersect f. Prove that H is 2-colourable if

$$8\sum_{j\geq k} \frac{d_{f,j}}{2^j} \leq 1$$

for each edge f of H.

(b) We say that a clause has *length* l if it consists of l distinct literals. By *$(\geq k)$-SAT*, we denote the class of all CNF-formulas in which each clause has length *at least* k. (The number of clauses and the lengths of all clauses are assumed to be finite.) Let $d_k, d_{k+1}, \ldots \in \mathbb{R}$ be such that $d_k + d_{k+1} + \cdots \leq 1$.

Suppose that $\mathcal{F}$ is an instance of $(\geq k)$-SAT such that

- for every $j \geq k$, each variable lies in at most $\frac{2^{j-2}d_j}{j}$ clauses of length j and
- for every $l > j \geq k$, each clause of length l contains at most j variables that appear in clauses of length j.

Prove that $\mathcal{F}$ is satisfiable.

Problem 4.3. Let G be a graph and let $d \geq 1$. Suppose that for every vertex v, there exists a list $S(v)$ of precisely $\lceil 2ed \rceil$ 'admissible' colours such that no colour in $S(v)$ is admissible for more than d neighbours of v. Prove that there is a 'proper' colouring of G (i.e. no two adjacent vertices have the same colour) assigning to each vertex an admissible colour.
Hint. The fewer vertices and colours play a role in the probability of a 'bad' event A, the simpler the expression for $\mathbb{P}[A]$ will be.

Problem 4.4. Let D be a directed graph without loops (i.e. $E(D)$ is a subset of $\{(u, v) \mid u, v \in V(D) \wedge u \neq v\}$) in which each vertex has precisely δ^+ many outgoing edges and at most Δ^- many ingoing edges. Suppose that k is a positive integer satisfying

$$e(\delta^+\Delta^- + 1)\left(1 - \frac{1}{k}\right)^{\delta^+} < 1.$$

Prove that there exists a colouring $c\colon V(D) \to \{0, \dots, k-1\}$ such that each vertex $v \in V(D)$ has an outgoing edge (v, w) with $c(w) \equiv c(v) + 1 \bmod k$.
Derive from this that if each vertex of D has *at least* δ^+ outgoing and at most Δ^- ingoing edges, then D contains a directed cycle whose length is a multiple of k.

Problem 4.5. Define the set $S \subset \mathbb{N}$ by letting each number n be in S with probability $1/2$ independently.

(a) For $k, l \in \mathbb{N}$, we set

$$w_l(k) = \left\lceil \frac{\ln(kl2^{k-1})}{\ln 2} \right\rceil.$$

Denote by A_l the event that there is a $k \geq 2$ such that S contains an arithmetic progression of the form

$$k - b, k, k + b, \dots, k + (w_l(k) - 2)b.$$

Prove that $\mathbb{P}[A_l] \leq 1/l$ and deduce from this that with probability 1, S does not contain an arithmetic progression of infinite length.

(b) Prove that

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{|S \cap [n]|}{n} = \frac{1}{2}\right] = 1.$$

To that end, for fixed $\varepsilon > 0$ and n, use the Chernoff bounds to find an upper bound for

$$\mathbb{P}\left[\left|\frac{|S \cap [n]|}{n} - \frac{1}{2}\right| \geq \varepsilon\right]$$

and apply a union bound to show that

$$\mathbb{P}\left[\exists n \geq n_0 \text{ with } \left|\frac{|S \cap [n]|}{n} - \frac{1}{2}\right| \geq \varepsilon\right] = o(n_0).$$

Where does this strategy fail when we use Chebyshev's inequality instead of Chernoff bounds?

Note. Recall that Szemerédi's Theorem states that each $S \subset \mathbb{N}$ with

$$\limsup_{n\to\infty} \frac{|S \cap [n]|}{n} > 0$$

contains infinitely many arithmetic progressions of length k for *every* k. Thus, the random set S above shows that we cannot expect an arithmetic progression of infinite length.