Probabilistic method in combinatorics and algorithmics
WS 2017/18

Exercise sheet 6

Exercises for the exercise session on 16/1/2018

Problem 6.1. Recall that in the proof of Azuma's inequality (Theorem 5.4), we defined the function

$$h(x) := \frac{e^{\alpha} + e^{-\alpha}}{2} + \frac{e^{\alpha} - e^{-\alpha}}{2} x$$

for some value $\alpha > 0$.

(a) Show that $\frac{e^{\alpha}+e^{-\alpha}}{2} < e^{\alpha^2/2}$ and $e^{\alpha x} \leq h(x)$ for all $x \in [-1, 1]$. (These were the missing links in the proof of Theorem 5.4.)

Also argue why Azuma's inequality implies that

$$\mathbb{P}\left[X_m \leq -s\sqrt{m}\right] < \exp\left(-\frac{s^2}{2}\right)$$

holds as well.

Hint. Comparing Taylor series might be the easiest way for the first part.

(b) Let $X_0, \ldots, X_m$ be a martingale and suppose that there are positive constants $c_1, \ldots, c_m$ such that

$$|X_i - X_{i-1}| \leq c_i$$

for $i = 1, \ldots, m$. Follow the lines of the proof of Theorem 5.4 to prove that for every $t > 0$,

$$\mathbb{P}\left[X_m \geq X_0 + t\right] < \exp\left(-\frac{t^2}{2\sum_{i=1}^{m} c_i^2}\right).$$

Hint. Define $Y_i = \frac{1}{c_i}(X_i - X_{i-1})$ in order to have $Y_i \in [-1, 1]$. Start with an arbitrary (positive) α and determine its value at the end of the proof. Then we need to bound

$$\mathbb{E}\left[e^{\alpha c_i Y_i} \mid X_0, \ldots, X_{i-1}\right].$$

To that end, the function h has to be defined slightly differently.

Problem 6.2. Suppose that an urn contains one red ball and one blue ball. A ball is drawn from the urn uniformly at random. After that, the ball is put back into the urn and another ball of the same colour is added to the urn. This process is repeated n times. Denote by X_n the proportion of red balls in the urn after these n steps (i.e. number of red balls divided by total number of balls). Use Azuma's inequality to prove that

$$\mathbb{P}\left[\left|X_n - \frac{1}{2}\right| \geq \varepsilon\right] < 2\exp\left(-\frac{6\varepsilon^2}{2\pi^2 - 15}\right).$$

Problem 6.3. Let $S_1, \dots, S_m$ be finite sets. Consider an arbitrary probability distribution $\mathbb{P}$ on the set

$$\Omega := \{(s_1, \dots, s_m) \mid \forall 1 \le i \le m \colon s_i \in S_i\}.$$

Let $f \colon \Omega \to \mathbb{R}$ be a function. For every $\sigma = (s_1, \dots, s_m) \in \Omega$, we choose $\tau = (t_1, \dots, t_m) \in \Omega$ according to $\mathbb{P}$ and set, for $i = 0, \dots, m$,

$$X_i(\sigma) := \mathbb{E}[f(\tau) \mid \forall 1 \le j \le i \colon s_j = t_j].$$

Then X_i is a random variable on Ω.

(a) Prove that $X_0, \dots, X_m$ is a martingale. Deduce from this that in particular, the edge exposure martingale and the vertex exposure martingale are indeed martingales.

(b) Prove that if $|f(\sigma) - f(\sigma')| \le 1$ holds for all σ, σ' that differ in only one coordinate, then we have
$$|X_i - X_{i-1}| \le 1$$
for all $i = 1, \dots, m$.

Problem 6.4. For an integer $n \ge 1$, let G be the graph with vertex set $V(G) = (\mathbb{Z}_7)^n$ and with $\{u, v\} \in E(G)$ if and only if u and v differ in only one coordinate. Suppose that $U \subset V(G)$ with $|U| = 7^{n-1}$ is given. For every $c > 0$, we define W_c to be the set of vertices of G with distance at least $(2 + c)\sqrt{n}$ from U. Show that

$$|W_c| < 7^n e^{-\frac{c^2}{2}}.$$

Hint. Define a martingale $X_0, \dots, X_n$ for which X_0 is the average (taken over all vertices of G) distance from U and $X_n(v)$ is the distance of v from U. Apply Azuma's inequality twice: first to prove that X_0 is 'small' and then to deduce the desired upper bound for W_c.

Problem 6.5. Let $k, n \to \infty$ such that $k = (2 + o(1)) \log_2 n$. Set

$$\mu := \binom{n}{k} 2^{-\binom{k}{2}}.$$

Prove that

$$\binom{n}{k} \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{-\binom{k}{2}} 2^{-\left(\binom{k}{2} - \binom{i}{2}\right)} = (1 + o(1)) \frac{k^4}{n^2} \mu^2$$

as $n \to \infty$. (This was used in the proof of Lemma 5.14.)
Hint. Remember that k is small compared to n, and we can choose n as large as necessary. If n becomes large, is there one summand that is much larger than all the others (even than all the others together)?