Homepage

`https://www.math.tugraz.at/comb/lehre`
`/1819/AAGT/AAGT.html`

Grading – Lecture

- Oral exams, dates by appointment

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- 30 – 45 min

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- 30 – 45 min
- General ideas are more important than details

Grading – Exercises

- Beginning of session: announce which problems you solved

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- Points per session: $6 \cdot \frac{\#\text{ solved problems}}{\#\text{ total problems}}$
- $S :=$ sum of best 5 sessions
- Presentation of solution at the board: 0 – 5 points
- $B :=$ sum of best 2 presentations

Grading – Exercises

$S + B$ max. 40

- ≥ 20: “genügend (4)”
- ≥ 25: “befriedigend (3)”
- ≥ 30: “gut (2)”
- ≥ 35: “sehr gut (1)”

Basic notations

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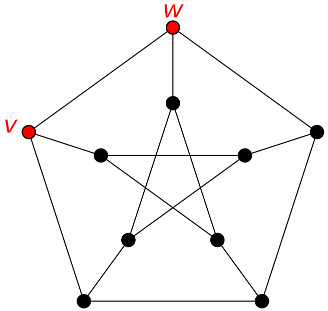
All graphs finite.

Adjajencies and incidencies

Definition

$v, w \in V(G)$

- v, w adjacent (neighbours) $:\Longleftrightarrow vw \in E(G)$;



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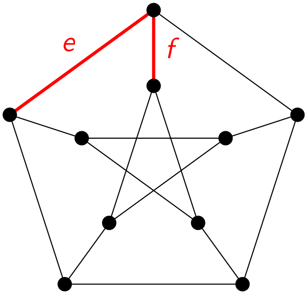

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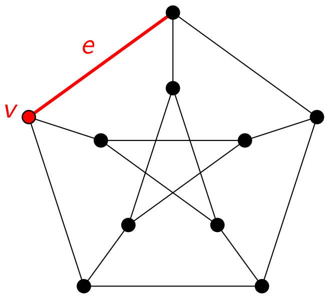
- v, e incident $:\Longleftrightarrow v \in e$.



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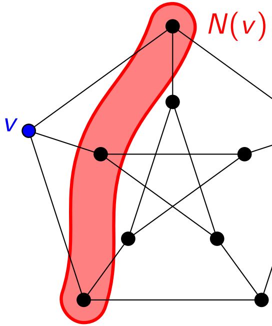

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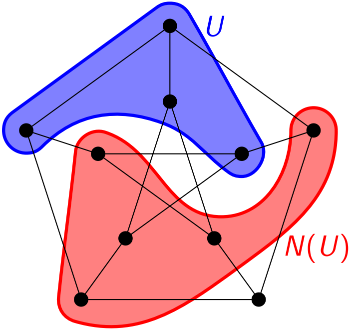
- neighbourhood of U: $N(U) := \left(\bigcup_{u \in U} N(u)\right) \setminus U$.



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- cubic $=$ 3-regular.

Degrees

Proposition

- $\delta(G) \leq d(G) \leq \Delta(G)$;
- $d(G) = \frac{2\|G\|}{|G|}$;
- $\#$ *vx's with odd degrees is even.*

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Proposition

- $\delta(G) \leq d(G) \leq \Delta(G)$;
- $d(G) = \frac{2\|G\|}{|G|}$;
- # *vx's with odd degrees is even.*

Proof.

Each edge is counted twice in $\sum d(v)$. □

Isomorphisms

Definition

G, H isomorphic $:\iff \exists$ bijection $f\colon V(G) \to V(H)$ s.t.

$$\forall u, v \in V(G)\colon uv \in E(G) \iff f(u)f(v) \in E(H).$$

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Our graphs: up to isomorphisms.

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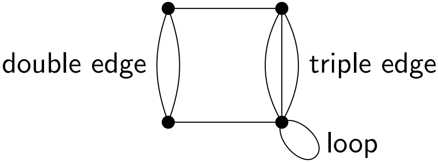

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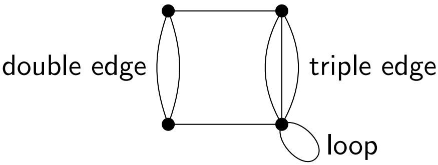


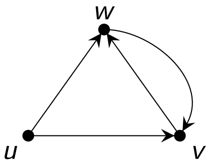
Definition

$d(v) :=$ # incident edges, loops counted twice $\geq |N(v)|$.

Directed graphs

Definition

Directed graph: $D = (V, E)$, where $E \subseteq V^2 \setminus \{(v, v) \mid v \in V\}$.



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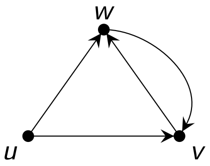

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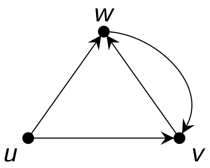

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- (u, v) = edge from u to v;
- indegree of v: $d^-(v) := \#$ edges to v;
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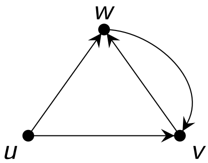
- min/av./max degrees $\delta^-(D)$, $\delta^+(D)$, $d^-(D)$, $d^+(D)$, $\Delta^-(D)$, and $\Delta^+(D)$.



Weighted graphs

Definition

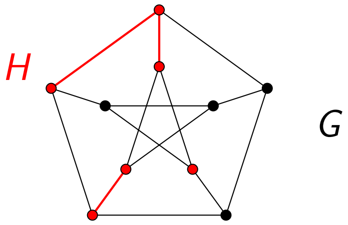
Weighted graph: Graph G with function $f\colon E(G) \to \mathbb{R}$.

- $f(e)$ = weight of e.

Subgraphs

Definition

- H subgraph of G $:\Longleftrightarrow$ $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; (Notation: $H \subseteq G$)



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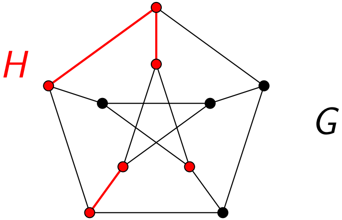

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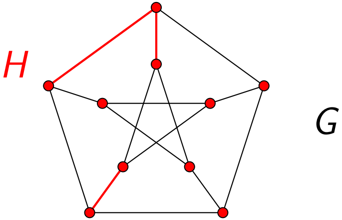

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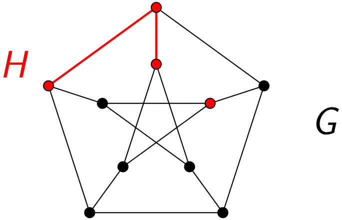
- H spanning subgraph $:\Longleftrightarrow$ $H \subseteq G$ and $V(H) = V(G)$;
- H induced subgraph $:\Longleftrightarrow$ $H \subseteq G$ and $E(H) = E(G) \cap \binom{V(H)}{2}$.



Subgraphs

Definition

$H_1, H_2 \subseteq G$

- $H_1 \cup H_2 := (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$;
- $H_1 \cap H_2 := (V(H_1) \cap V(H_2), E(H_1) \cap E(H_2))$.

Paths

Definition

- Path of length n: graph P with $V(P) = \{v_0, \ldots, v_n\}$ and $v_i v_j \in E(P)$ iff $i - j = \pm 1$. (Notation: $P = v_0 \ldots v_n$)

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- A–B path ($A, B \subseteq V(G)$): path $P = v_0 \dots v_n$ in G with $V(P) \cap A = \{v_0\}$ and $V(P) \cap B = \{v_n\}$.

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- H-path ($H \subseteq G$): path $v_0 \dots v_n$ in G with $n \geq 1$ and $H \cap P = \{v_0, v_n\}$.

Cycles and complete graphs

Definition

- Cycle of length n ($n \geq 3$): graph C with $V(C) = \{v_1, \ldots, v_n\}$ and $v_i v_j \in E(C)$ iff $i - j = \pm 1 \bmod n$.

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- Complete graph K^n: $|V(K^n)| = n$, $E(K^n) = \binom{V(K^n)}{2}$.

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- $G + F := (V(G), E(G) \cup F)$.

Short notation

Abbreviation if $A = \{a\}$, $B = \{b\}$, $U = \{u\}$, or $F = \{e\}$:

a–B path, A–b path, a–b path, $G - v$, $G - e$, $G + e$.

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Proof.

Exercise. □

Contractions

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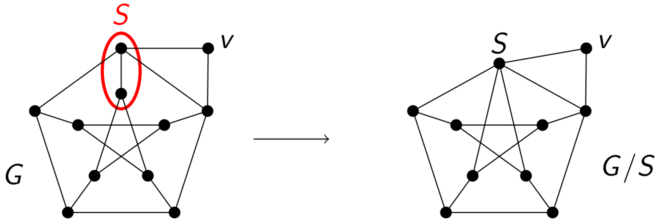
Contraction G/S (for $S \subseteq V(G)$):

$$V(G/S) := V(G - S) \cup \{S\},$$
$$E(G/S) := E(G - S) \cup \{vS \mid \exists s \in S \colon vs \in E(G)\}$$



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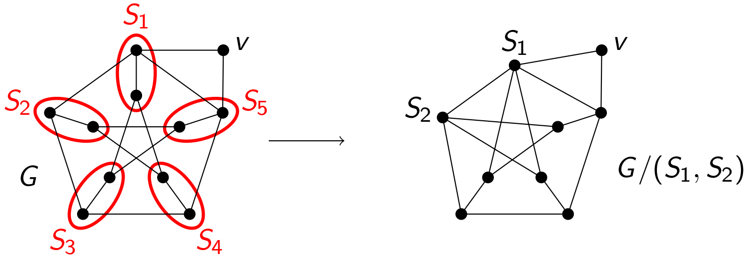
$$G/(S_1, \ldots, S_k) := \Big(((G/S_1)/\ldots)/S_k\Big).$$



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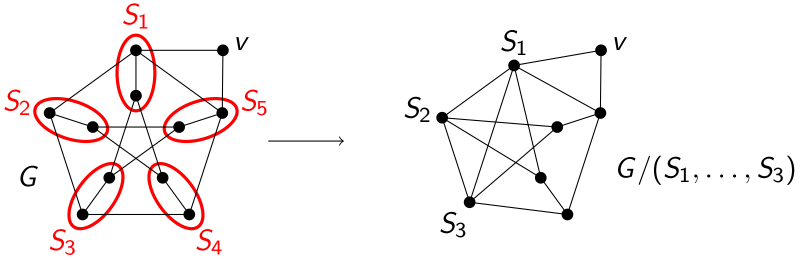

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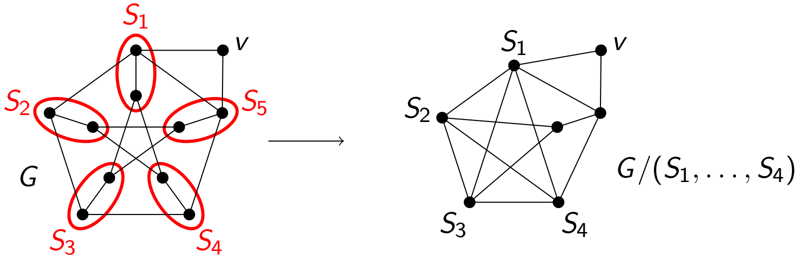

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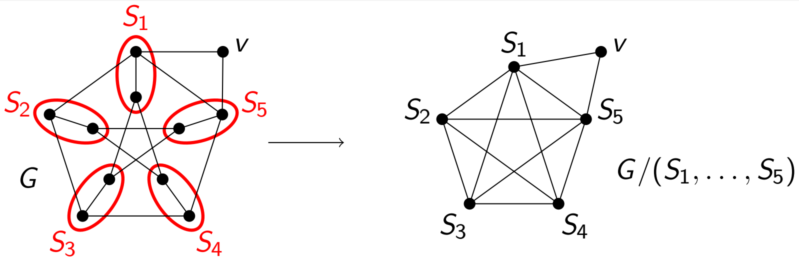

Minors

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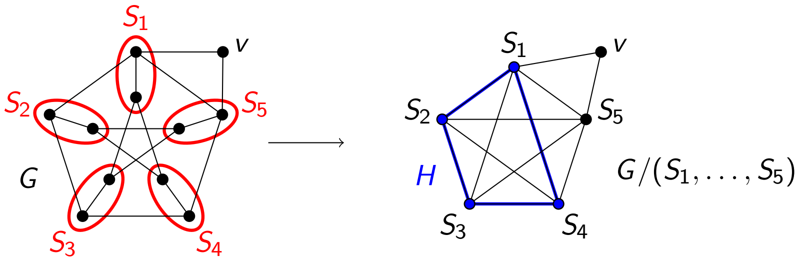

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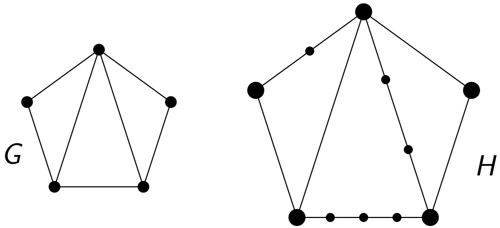
- H subdivision of G: edges of $G \longrightarrow$ paths of length ≥ 1.



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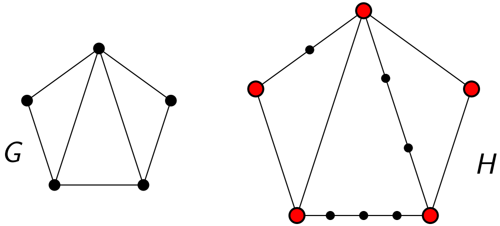
- Branch vx of subdivision H: vx of G in H.



Definition

- H minor of G: $H \subseteq G/(S_1, \ldots, S_k)$ for disjoint, connected branch sets $S_1, \ldots, S_k$.
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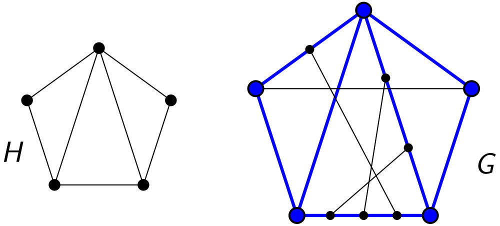
- Branch vx of subdivision H: vx of G in H.
- H topological minor of G: G contains subdivision of H.



Algorithms and running time

Abbreviation for running time: $n = |G|$, $m = \|G\|$.

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$f, g : \mathbb{N} \to \mathbb{R}_{>0}$.

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- $f(n) = \Omega(g(n))$: $\exists c^-, N > 0$ s.t. $f(n) \ge c^- g(n)$ $\forall n \ge N$;
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Mostly: running time $f(n) = O(g(n))$ (worst case analysis).