Analytic combinatorics

SS 2019

Exercise sheet 2

Exercises for the exercise session on 27 March 2019

In all of the following problems, you might end up with a sum expression for the coefficient. If you see a way to simplify this sum expression to a closed formula (such as writing 2^n instead of the sum formula $\sum_{k=0}^{n}\binom{n}{k}$), please do so. (But don't spend too much time looking for a way to simplify sums.)

Problem 2.1. Consider the number of ways a string of n identical letters, say x, can be 'bracketed'. The rule is best stated recursively: x itself is a bracketing and if $\sigma_1, \sigma_2, \dots, \sigma_k$ with $k \geq 2$ are bracketed expressions, then the k-ary product $(\sigma_1\sigma_2\cdots\sigma_k)$ is a bracketing as well. For instance, $(((xx)x(xxx))((xx)(xx)x))$ is a bracketing of 11 letters. Let $\mathcal{S}$ denote the class of all bracketings, where size is taken to be the number of instances of x.

(a) Derive a recursive formula for $s_n := |\mathcal{S}_n|$ and derive the corresponding expression in terms of generating functions.

(b) Determine s_n via the Lagrange Inversion Theorem.

Problem 2.2. The class of mappings from $\{1, \dots, n\}$ to itself has the exponential generating function

$$F(z) = \frac{1}{1 - C(z)},$$

where $C(z)$ is the exponential generating function of the class of Cayley trees and thus satisfies $C(z) = ze^{C(z)}$. Determine the coefficient

$$[z^n]F(z)$$

by using the Lagrange formula that if $A(z) = z\phi(A(z))$, then

$$[z^n]\psi(A(z)) = \frac{1}{n}[u^{n-1}]\left(\psi'(u)(\phi(u))^n\right).$$

Problem 2.3. The class of mappings from $\{1, \dots, n\}$ to itself without fixed points has the exponential generating function

$$G(z) = \frac{e^{-C(z)}}{1 - C(z)},$$

where $C(z)$ is the exponential generating function of Cayley trees. Determine the coefficient

$$[z^n]G(z).$$

Problem 2.4. The bivariate exponential generating function of permutations counted according to both the number of elements (marked by z) and the number of cycles (marked by u) is

$$P(z, u) = (1 - z)^{-u} = \sum_{n=0}^{\infty} \binom{u + n - 1}{n} z^n.$$

Denote by X_n the number of cycles in a random permutation, which is chosen uniformly at random from all permutations of size n. Determine the second factorial moment

$$\mathbb{E}\left(X_n(X_n - 1)\right).$$

Problem 2.5. Let C_n be a Cayley tree, chosen uniformly at random from all Cayley trees on n vertices. Determine the expected degree of the root of C_n.