Analytic combinatorics

SS 2019

TU Graz

Exercise sheet 3

Exercises for the exercise session on 8 April 2019

Problem 3.1. Let $R > 0$ and let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that

- $f(0) = 0$,
- $f'(0) \neq 0$,
- $f(z) \neq 0$ for all z with $0 < |z| < R$.

Using only the rule of l'Hospital, prove that the function

$$g(z) = \frac{1}{f(z)} - \frac{1}{f'(0)z}$$

is holomorphic on the open disc of radius R around the origin.

Problem 3.2. Let $B(z) = \sum_n B_n z^n$ denote the ordinary generating function for the class of binary strings with no consecutive 0's (note: the empty string is included in this class). Derive

(a) a closed expression for $B(z)$,

(b) an asymptotic expression for B_n.

Problem 3.3. An *alignment* is a sequence of cycles. Let $A(z) = \sum_n A_n \frac{z^n}{n!}$ denote the exponential generating function for the class of alignments. Derive

(a) a closed expression for $A(z)$,

(b) an asymptotic expression for A_n.

Problem 3.4. Let G_n denote the number of vertex-labelled 2-regular (simple) graphs on vertex set $[n]$, all whose components have even size. Derive

(a) a closed expression for $G(z) = \sum_n G_n \frac{z^n}{n!}$,

(b) an asymptotic expression for P_n.

Problem 3.5. Let $P(z) = \sum_n P_n \frac{z^n}{n!}$ denote the exponential generating function for the class of permutations consisting of cycles whose lengths are multiples of three. Derive

(a) a closed expression for $P(z)$,

(b) an asymptotic expression for P_n.