Analytic combinatorics
SS 2019

Exercise sheet 4

Exercises for the exercise session on 13 May 2019

Problem 4.1. Let $R(z), U(z)$ denote the (ordinary) generating functions of r-nary trees (for $r \geq 2$) and unary-binary trees, respectively. (See also Problems 1.1(a) and 1.2.) Use the singular inversion theorem to determine asymptotic formulae for $[z^n]R(z)$ and $[z^n]U(z)$.

Problem 4.2. For $r \geq 2$, let $\mathcal{C}_r$ be the class of Cayley trees in which every vertex has at most r children. Denote by $C_r(z)$ the exponential generating function of $\mathcal{C}_r$.

(a) Show that the singular inversion theorem can be applied to $C_r(z)$ for every r and determine an asymptotic formula for $[z^n]C_r(z)$ in the special case $r = 2$.

(b) Prove that the dominant singularity ρ_r of $C_r(z)$ converges to $\frac{1}{e}$ for $r \to \infty$.

Problem 4.3. The ordinary generating function $T(z)$ of the class of triangulations of convex polygons satisfies

$$T(z) = 1 + zT(z)^2. \tag{1}$$

Show that the implicit function scheme *cannot* be applied to the function $G(z, w) = 1+zw^2$ arising from (1). Define an auxilary function $\tilde{T}(z)$ to which the implicit function scheme applies and use this to derive asymptotic formulae for $[z^n]\tilde{T}(z)$ and for $[z^n]T(z)$.

Problem 4.4. A *rooted dissection* of a convex polygon with a distinguished edge (the *root*) is a set of non-crossing diagonals of the polygon. Let $\mathcal{D}_n$ be the class of rooted dissections of regular $(n+2)$-gons. The ordinary generating function $D(z)$ of $\mathcal{D} = \bigcup_n \mathcal{D}_n$ satisfies

$$D(z) = \frac{z(1 + D(z))^2}{1 - z(1 + D(z))}.$$

Use the implicit function scheme to derive an asymptotic formula for $[z^n]D(z)$.

Problem 4.5. Let T be a tree with at least three vertices and let v be a vertex of T. Delete v from T and denote the components of the resulting forest by $C_1, \ldots, C_k$. For each $i = 1, \ldots, k$, let w_i be the neighbour of v in C_i and denote by u_i a vertex in C_i with $d(u_i, v)$ (measured in T) maximal. Without loss of generality, suppose that $d(u_1, v) \geq d(u_2, v) \geq \cdots \geq d(u_k, v)$. Prove that

(a) if $k = 1$ or $d(u_1, v) \geq d(u_2, v) + 2$, then $v \notin c(T)$;

(b) if $k \geq 2$ and $d(u_1, v) = d(u_2, v)$, then $c(T) = \{v\}$;

(c) if $k \geq 2$ and $d(u_1, v) = d(u_2, v) + 1$, then $c(T) = \{v, w_1\}$.