Analytic combinatorics
SS 2019

Exercise sheet 5

Exercises for the exercise session on 22 May 2019

Note. In the lecture, the tree dissymmetry theorem was phrased for the class of all *trees. More generally, it is true for any class of unrooted trees. (Why?)*

Problem 5.1. Denote by $B(z)$ the exponential generating function of the class of rooted labelled trees in which every vertex has either no children or precisely two children (i.e. binary trees with labels, but without the embedding into the plane). Determine

(a) the dominant singularities of $B(z)$,

(b) singular expansions of $B(z)$ near its dominant singularities,

(c) an asymptotic expression for $[z^n]B(z)$ for n odd (otherwise the coefficient is zero).

Problem 5.2. Denote by $T(z)$ the exponential generating function of the class of labelled (unrooted) trees in which all vertices have degree 1 or 3. (Observe that every such tree has even order.)

(a) Apply the dissymmetry theorem to derive an expression of $T(z)$ in terms of $T^{\circ-\circ}(z)$.

(b) Determine the dominant singularities of $T^{\circ-\circ}(z)$ and singular expansions near those singularities.

(c) Use (a) and (b) to determine an asymptotic formula for $[z^n]T(z)$ for n even.

Problem 5.3. Let $B(z)$ and $T(z)$ be as in Problems 5.1 and 5.2.

(a) Apply the dissymmetry theorem to derive an expression of $T(z)$ in terms of $B(z)$.

(b) Use (a) to determine an asymptotic formula for $[z^n]T(z)$ for n even.

Problem 5.4. Let $P(z)$ be the ordinary generating function of the class of unrooted, unlabelled plane trees. Apply the dissymmetry theorem to determine

(a) the dominant singularities of $P(z)$,

(b) derive a singular expansion of $P(z)$ near those singularities,

(c) an asymptotic expression for $[z^n]P(z)$.

Problem 5.5. For $n \geq 1$, derive upper bounds for

$$\frac{1}{n!} = [z^n]e^z \qquad \text{and} \qquad \binom{2n}{n} = [z^n](1+z)^{2n}$$

(a) by applying the saddle-point bounds;

(b) by applying (if possible) the upper bound provided by the solution of the saddle-point equation.

Which radius r yields the strongest bounds in (a)? If (b) is applicable, compare the bound with the bound from (a). Which one is stronger?