Exercise sheet 6

Exercises for the exercise session on 5 June 2019

Problem 6.1. Suppose that $G(z), H(z)$ are both Hayman admissible with the same radius of convergence ρ. Prove that the function $F(z) := G(z)H(z)$ satisfies the capture condition of Hayman admissibility and that there exist $\rho_0 \in (0, \rho)$ and functions

$$\theta_0^-, \theta_0^+ \colon (\rho_0, \rho) \to (0, \pi)$$

such that $F(z)$ satisfies the locality condition for $|\theta| \leq \theta_0^-$ and the decay condition for $|\theta| > \theta_0^+$. (Meta question: Where does the difficulty lie in finding a single function θ_0 that works for both conditions?)

Problem 6.2. Prove that $F(z) = \frac{1}{1-z}$ is *not* Hayman admissible. Furthermore, show that $G(z) = e^{\frac{z}{1-z}}$ the capture condition and determine what functions θ_0 would satisfy the necessary condition $b(s)^{-1/2} \ll \theta_0 \ll c(s)^{-1/3}$ for the locality condition and the decay condition.
Under the assumption that $G(z)$ is indeed Hayman admissible, determine an asymptotic expression for the coefficients $[z^n]G(z)$.

Problem 6.3. For $k \geq 2$, consider the class $\mathcal{P}_k$ of permutations consisting only of cycles of length at most k, and let $P_k(z)$ denote its exponential generating function. Determine an asymptotic formula for $[z^n]P_k(z)$ using saddle-point asymptotics.

Problem 6.4. Denote by $P(z)$ the ordinary generating function of plane rooted trees, and let $F(z) := e^{P(z)}$. Use saddle-point estimates of large powers to determine asymptotic formulae for $[z^n]P(z)$ and $[z^n]F(z)$.

Problem 6.5. Denote by $S(z)$ the ordinary generating function of bracketings (see also Problem 2.1). Apply the standard Lagrangean framework to derive an asymptotic expression for $[z^n]S(z)$.