Probabilistic method in combinatorics and algorithmics
WS 2018/19

Exercise sheet 1

Exercises for the exercise session on 11 Oct. 2018

Problem 1.1. Prove that given any n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with $||\mathbf{v}_i|| = 1$ for all i, there exist n numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{+1, -1\}$ such that

$$\left|\left|\sum_{i=1}^{n} \epsilon_i \mathbf{v}_i\right|\right| \leq \sqrt{n}.$$

Problem 1.2. A *tournament* is an orientation of a complete graph, i.e. for every pair of distinct vertices v, w, exactly one of the directed edges (v, w) and (w, v) is present. A *Hamiltonian path* in a tournament is a directed path passing through all vertices.

(a) Prove that there exists a tournament on n vertices that has at least $n!\ 2^{-n+1}$ Hamiltonian paths.

(b) Prove that for k satisfying

$$\binom{n}{k}\left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k} < 1,$$

there is a tournament on n vertices with the property that for every set of k players, there is some player who beats them all.

Problem 1.3. A *k-uniform hypergraph* is a pair $H = (V, E)$ with vertex set V and (hyper)edge set E, where every edge is a subset of V containing exactly k elements.

(a) Prove that if a k-uniform hypergraph $H = (V, E)$ has fewer than 2^{k-1} edges, then there exists a colouring of V by two colours so that no edge is monochromatic.

(b) Prove that if a k-uniform hypergraph $H = (V, E)$ has fewer than $\frac{4^{k-1}}{3^k}$ edges, then there exists a colouring of V by four colours so that all four colours are represented in every edge.