Probabilistic method in combinatorics and algorithmics
WS 2018/19

Exercise sheet 3

Exercises for the exercise session on 8 Nov. 2018

Problem 3.1. Let H be a fixed graph with k vertices and $m \geq 1$ edges. We define the *maximum density* d_H of H by

$$d_H := \max\left\{ \frac{|E(H')|}{|V(H')|} \;\middle|\; H' \text{ is a non-empty subgraph of } H \right\}.$$

Prove that

$$\mathbb{P}[H \text{ is a subgraph of } G(n,p)] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = o\left(n^{-\frac{1}{d_H}}\right), \\ 1 & \text{if } p = \omega\left(n^{-\frac{1}{d_H}}\right). \end{cases}$$

(Note that for a given set $S \in \binom{[n]}{k}$, there might be more than one bijection $V(H) \to S$ that preserves adjacencies. At some point in your proof, you should show that the number of such bijections does not affect your arguments.)

Problem 3.2. Let $H = (V, E)$ be a hypergraph in which every edge has at least k-elements. Suppose that each edge of H intersects at most $d \geq 1$ other edges. Prove that if $e(d+1)2^{1-k} \leq 1$, then H is 2-colourable.

(Recall that H is called *2-colourable* if there exists a colouring of V by two colours so that no edge in E is monochromatic.)

Problem 3.3. The *Ramsey number* $R(t)$ is defined as the smallest integer n such that any graph G on n vertices contains either a clique of order t or an independent set of order t.

(a) Prove that if $e\left(\binom{t}{2}\binom{n-2}{t-2} + 1\right) 2^{1-\binom{t}{2}} \leq 1$, then $R(t) > n$.

(b) Prove that when $t \to \infty$,

$$R(t) \geq \frac{\sqrt{2}}{e}(1 + o(1))\, t\, 2^{t/2}.$$