Probabilistic method in combinatorics and algorithmics
WS 2018/19

Exercise sheet 4

Exercises for the exercise session on 19 Nov. 2018

Problem 4.1. Prove the following variant of Chernoff bounds.
Suppose that $X_1, \ldots, X_n$ are independent random variables, where each X_i has only a finite number of possible values $x_{i,1}, \ldots, x_{i,m_i} \in [-1, 1]$ and satisfies $\mathbb{E}[X_i] = 0$. Let $X = X_1 + \cdots + X_n$. Then for $0 \le t \le 2\mathrm{Var}[X]$,

$$\mathbb{P}[X \ge t] \le \exp\left(-\frac{t^2}{4\mathrm{Var}[X]}\right).$$

(*Hint.* Start as in the proof of the Chernoff bound 1 from the lecture. Prove that $\mathbb{E}[\exp(uX_i)]$ is bounded by $1 + u^2\mathrm{Var}[X_i]$ if $u \le 1$.)

Problem 4.2. Let $p = p(n) \in (0, 1)$ be given.

(a) For $t > 0$ and a fixed vertex v of $G(n, p)$, compare the bounds on $\mathbb{P}[|d(v) - \mathbb{E}[d(v)]| \ge t]$ provided by Chebyshev's inequality and by the Chernoff bounds 1 and 2. In each of the three cases, how large does t have to be in order to deduce that

$$\mathbb{P}[|d(v) - \mathbb{E}[d(v)]| \ge t] = o(1)?$$

(b) How large does t have to be if we want to prove that

$$\mathbb{P}[\exists v \in [n]\colon\ |d(v) - \mathbb{E}[d(v)]| \ge t] = o(1)? \tag{1}$$

(c) Are there functions $p(n)$ for which the minimum requirements for t in (1) from the two Chernoff bounds coincide?

Problem 4.3. Suppose we place n balls in n bins, where each ball chooses its bin uniformly at random and independently from the other balls.

(a) Prove that for each $\varepsilon > 0$,

$$\mathbb{P}\left[\exists \text{ a bin with at least } \left(\frac{3}{2} + \varepsilon\right) \ln n \text{ balls}\right] = o(1).$$

(b) By how much can we improve the lower bound on the number of balls in a bin?

(c) If we have n^2 balls in total, for what k can we prove that

$$\mathbb{P}\left[\exists \text{ a bin with at least } k \text{ balls}\right] = o(1)?$$