Probabilistic method in combinatorics and algorithmics
WS 2018/19

Exercise sheet 5

Exercises for the exercise session on 29 Nov. 2018

Problem 5.1. Let X and Y be random variables.

(a) Prove that
$$\mathbb{E}\Big[f(X)g(Y)\Big] = \mathbb{E}\Big[f(X)\ \mathbb{E}[g(Y)|X]\Big]$$
if f and g are real-valued functions.

(b) Prove that
$$\mathrm{Var}[aX + bY] \leq a\mathrm{Var}[X] + b\mathrm{Var}[Y]$$
if X and Y are real-valued and independent and $a, b \in [0, 1]$.

Problem 5.2. Given a constant $\alpha \in \mathbb{R}$, let the function $h(x)$ be defined by
$$h(x) := \frac{e^{\alpha} + e^{-\alpha}}{2} + \frac{e^{\alpha} - e^{-\alpha}}{2}x.$$

(a) Prove that
$$e^{\alpha x} \leq h(x)$$
for all $x \in [-1, 1]$.

(b) Prove that
$$\frac{e^{\alpha} + e^{-\alpha}}{2} \leq e^{\frac{\alpha^2}{2}}.$$

Problem 5.3. Let $(X_0, X_1, \ldots, X_m)$ be a martingale with $X_0 = \mathbb{E}[X_m]$ and
$$|X_i - X_{i-1}| \ \leq\ c_i \quad \textit{for all} \quad 1 \leq i \leq m.$$

Prove that for any $t > 0$,
$$\mathbb{P}\left[X_m > \mathbb{E}[X_m] + t\right] \leq \exp\left(-\frac{t^2}{2\sum_{1\leq i\leq m} c_i^2}\right)$$
and
$$\mathbb{P}\left[X_m < \mathbb{E}[X_m] - t\right] \leq \exp\left(-\frac{t^2}{2\sum_{1\leq i\leq m} c_i^2}\right).$$