Probabilistic method in combinatorics and algorithmics
WS 2018/19

Exercise sheet 6

Exercises for the exercise session on 10 Dec. 2018

Problem 6.1. Let $p_1, p_2, \ldots, p_8 \in [0,1]$ satisfy $\sum_{i=1}^{8} p_i = 1$, let the random variables Y_0, Y_1, and Y_2 be defined by

$$\begin{aligned} \mathbb{P}[Y_0 = -1, Y_1 = -1, Y_2 = -1] &= p_1, \\ \mathbb{P}[Y_0 = -1, Y_1 = -1, Y_2 = 1] &= p_2, \\ \mathbb{P}[Y_0 = -1, Y_1 = 1, Y_2 = -1] &= p_3, \\ \mathbb{P}[Y_0 = 1, Y_1 = -1, Y_2 = -1] &= p_4, \\ \mathbb{P}[Y_0 = -1, Y_1 = 1, Y_2 = 1] &= p_5, \\ \mathbb{P}[Y_0 = 1, Y_1 = -1, Y_2 = 1] &= p_6, \\ \mathbb{P}[Y_0 = 1, Y_1 = 1, Y_2 = -1] &= p_7, \\ \text{and } \mathbb{P}[Y_0 = 1, Y_1 = 1, Y_2 = 1] &= p_8, \end{aligned}$$

and let the random variables X_0, X_1, and X_2 be defined by $X_i = \sum_{j \le i} Y_j$. Find values for $p_1, p_2, \ldots, p_8$ so that we simultaneously have

$$\mathbb{E}[X_1 | X_0 = x] = x \text{ for all } x,$$

$$\mathbb{E}[X_2 | X_1 = x] = x \text{ for all } x,$$

and (X_0, X_1, X_2) is **not** a martingale.

Problem 6.2. Let $p = n^{-\alpha}$ for a fixed $\alpha > \frac{1}{2}$. Show

$$\mathbb{P}[G(n,p) \text{ is triangle-free}] = \exp\left(-(1+o(1))\frac{n^{3-3\alpha}}{6}\right).$$

Problem 6.3. Let $p = n^{-\alpha}$ for a fixed $\alpha \le \frac{1}{2}$. Show

$$\mathbb{P}[G(n,p) \text{ is triangle-free}] \le \exp\left(-(1+o(1))\frac{n^{2-\alpha}}{36}\right).$$