Analytic combinatorics
SS 2020

Exercise sheet 1

Exercises for the exercise session on 11 March 2020

Problem 1.1. (a) Let $\mathcal{T}$ be the class of ternary trees and $\mathcal{T}_n$ the class of ternary trees of size n.

(i) Express $\mathcal{T}$ in terms of $\mathcal{T}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.

(ii) Derive a closed formula for $\mathcal{T}_n$ (using Lagrange Inversion Theorem).

(b) Let $\mathcal{U}$ be the class of unary-binary trees and $\mathcal{U}_n$ the class of unary-binary trees of size n.

(i) Express $\mathcal{U}$ in terms of $\mathcal{U}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.

(ii) Derive a closed formula for $\mathcal{U}_n$ (using Lagrange Inversion Theorem).

Problem 1.2. Consider the number of ways a string of n identical letters, say x, can be 'bracketed'. The rule is best stated recursively: x itself is a bracketing and if $\sigma_1, \sigma_2, \ldots, \sigma_k$ with $k \geq 2$ are bracketed expressions, then the k-ary product $(\sigma_1 \sigma_2 \cdots \sigma_k)$ is a bracketing. For instance: $(((xx)x(xxx))((xx)(xx)x))$. Let $\mathcal{S}$ denote the class of all bracketings, where size is taken to be the number of instances of x.

(a) Express $\mathcal{S}$ in terms of $\mathcal{S}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.

(b) Derive a recursive formula for $s_n := |\mathcal{S}_n|$ and derive the corresponding expression in terms of generating functions.

(c) Derive a closed formula for s_n and its asymptotic formula.

Problem 1.3. Consider a sequence of numbers $x = (x_0 = 0, x_1, \ldots, x_{2n-1}, x_{2n} = 0)$ satisfying $x_i \geq 0, \quad |x_i - x_{i-1}| = 1$ for $1 \leq i \leq 2n$. This represents an excursion that take place in the upper half-plane, also known as *Dyck paths of length* $2n$. Let $\mathcal{D}$ be the class of Dyck paths and $\mathcal{D}_{2n}$ the class of Dyck paths of length $2n$.

(a) Express $\mathcal{D}$ in terms of $\mathcal{D}$ and basic constructions (e.g. combinatorial sum, etc) and derive the corresponding expression in terms of generating functions.

(b) Derive a closed formula for $|\mathcal{D}_{2n}|$ and its asymptotic formula.

Problem 1.4. A *meander* is a word over $\{-1, +1\}$ such that the sum of the values of any of its prefixes is a non-negative integer. A *bridge* is a word over $\{-1, +1\}$ whose values of its letters sum to 0. Note that a meander represents a walk that wanders in the first quadrant, and a brige a walk that wanders above and below the horizontal line, but its final altitute is constrained to be 0. Let $\mathcal{M}$ be the class of meanders and $\mathcal{B}$ the class of bridges.

(a) Express $\mathcal{M}$ and $\mathcal{B}$ in terms of $\mathcal{D}$ and basic constructions and derive the corresponding expression in terms of generating functions.

(b) Derive a closed formula for $|\mathcal{M}_{2n}|$ and $|\mathcal{B}_{2n}|$ (i.e. the numbers of meanders and bridges of length $2n$, respectively).