Exercise sheet 2

Exercises for the exercise session on 29 April 2020

Problem 2.1. Recall from Problems 1.3 and 1.4 that the numbers of Dyck paths and of meanders of length $2n$ satisfy

$$|\mathcal{D}_{2n}| = \frac{1}{n+1}\binom{2n}{n} \qquad \text{and} \qquad |\mathcal{M}_{2n}| = \binom{2n}{n}.$$

Use these results to deduce a closed formula for $|\mathcal{M}_{2n+1}|$ and determine its asymptotic value.

Problem 2.2. Denote by $\mathcal{F}_n$ and $\mathcal{H}_n$ the classes of all mappings $[n] \to [n]$ and of all such mappings without fixed points, respectively. We know from the lecture that the exponential generating functions of $\mathcal{F} := \bigcup_n \mathcal{F}_n, \mathcal{H} := \bigcup_n \mathcal{H}_n$ satisfy

$$F(z) = \frac{1}{1 - C(z)} \qquad \text{and} \qquad H(z) = \frac{e^{-C(z)}}{1 - C(z)},$$

respectively, where $C(z)$ is the exponential generating function of Cayley trees. Determine $f_n := |\mathcal{F}_n|$ and $h_n := |\mathcal{H}_n|$

(a) directly (i.e. without using generating functions);

(b) by determining the coefficients $[z^n]F(z)$ and $[z^n]H(z)$ via Lagrange inversion.

Verify that (a) and (b) yield the same results.

Problem 2.3. Denote by $\mathcal{U}$ the class of labelled *unicyclic* graphs, that is, connected graphs with precisely one cycle.

(a) Express $\mathcal{U}$ in terms of the class $\mathcal{C}$ of Cayley trees and basic constructions and derive the corresponding expression in terms of exponential generating functions.

(b) Use Lagrange inversion to determine (as sum formula for) the number of unicyclic graphs on n vertices.

Problem 2.4. The bivariate exponential generating function of permutations counted according to both the number of elements (marked by z) and the number of cycles (marked by u) is

$$P(z, u) = (1 - z)^{-u} = \sum_{n=0}^{\infty} \binom{u + n - 1}{n} z^n.$$

Denote by X_n the number of cycles in a random permutation, which is chosen uniformly at random from all permutations of size n. We know from the lecture that

$$\mathbb{E}[X_n] = H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

Determine the second factorial moment

$$\mathbb{E}[X_n(X_n - 1)]$$

and deduce from the result that $\mathbb{V}[X_n] \le \mathbb{E}[X_n]$.