Exercise sheet 3

Exercises for the exercise session on 13 May 2020

Problem 3.1. Let C_n and P_n be a Cayley tree and a plane rooted tree, respectively, chosen uniformly at random from all Cayley trees or plane rooted trees on n vertices, respectively. Determine

(a) the expected degree of the root of C_n;

(b) the expected degree of the root of P_n.

Problem 3.2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function and suppose that there is a point $z_0 \in \mathbb{C} \setminus \{0\}$ and a real number $R > |z_0|$ such that

- $f(z_0) = 0$;
- $f(z) \neq 0$ for all $z \neq z_0$ with $|z| < R$;
- $f'(z_0) \neq 0$.

(a) Prove that there exists a function $H(z)$ that is holomorphic on the open disc of radius R around the origin and satisfies

$$\frac{1}{f(z)} = \frac{1}{f'(z_0)(z - z_0)} + H(z).$$

(This in particular proves the missing part in Example 4.4.3 from the lecture.)

(b) Derive an asymptotic expression for $[z^n]\frac{1}{f(z)}$.

Problem 3.3. Let $B(z) = \sum_n B_n z^n$ and $S(z) = \sum_n S_n z^n$ denote the ordinary generating functions for the classes of binary strings with no consecutive 0's (note: the empty string is included in this class) and the class of bracketings, respectively. Recall from Problem 1.2 that

$$S(z) = \frac{1 + z - \sqrt{z^2 - 6z + 1}}{4}.$$

Derive

(a) a closed expression for $B(z)$;

(b) an asymptotic expression for B_n;

(c) an asymptotic expression for S_n.

Problem 3.4. An *alignment* is a sequence of cycles. Denote by $A(z) = \sum_n A_n \frac{z^n}{n!}$ the exponential generating function for the class of alignments. Moreover, recall that the ordinary generating function $T(z) = \sum_n T_n z^n$ of triangulations is given by

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Derive

(a) a closed expression for $A(z)$;

(b) an asymptotic expression for A_n;

(c) an asymptotic expression for T_n, using singularity analysis. Compare the result with the asymptotic expression obtained by applying Stirling's formula to the closed expression for T_n.