Analytic combinatorics
SS 2020

Exercise sheet 4

Exercises for the exercise session on 25 May 2020

Problem 4.1. Let G_n denote the number of vertex-labelled 2-regular (simple) graphs on vertex set $[n]$, all whose components have even size.

(a) Derive a closed expression for the exponential generating function $G(z) = \sum_n G_n \frac{z^n}{n!}$.

(b) Use the transfer theorem for multiple singularities to derive an asymptotic expression for G_n. If the largest term in the asymptotic expression is $g(n)$, the error term should have order $O(\frac{g(n)}{n})$ (i.e. all terms that are smaller than $g(n)$, but of larger order than $\frac{g(n)}{n}$, should be stated explicitly).

Problem 4.2. For $r \geq 2$, let $\mathcal{C}_r$ be the class of Cayley trees in which every vertex has at most r children. Denote by $C_r(z)$ the exponential generating function of $\mathcal{C}_r$.

(a) Show that the singular inversion theorem can be applied to $C_r(z)$ for every r and determine an asymptotic formula for $[z^n]C_r(z)$ in the special case $r = 2$.

(b) Prove that the dominant singularity ρ_r of $C_r(z)$ converges to $\frac{1}{e}$ for $r \to \infty$.

Problem 4.3. The ordinary generating function $T(z)$ of the class of triangulations of convex polygons satisfies

$$T(z) = G(z, T(z)) := 1 + zT(z)^2.$$

Check the conditions of the implicit function scheme for the functions $T(z)$ and $G(z, w)$. Which are satisfied and which are not? Define an auxilary function $\tilde{T}(z)$ to which the implicit function scheme applies and use this to derive asymptotic formulae for $[z^n]\tilde{T}(z)$ and for $[z^n]T(z)$.

Problem 4.4. A *rooted dissection* of a convex polygon with a distinguished edge (the *root*) is a set of non-crossing diagonals of the polygon. Let $\mathcal{D}_n$ be the class of rooted dissections of regular $(n+2)$-gons. The ordinary generating function $D(z)$ of $\mathcal{D} = \bigcup_n \mathcal{D}_n$ satisfies

$$D(z) = (1 + D(z)) \left(\frac{1}{1 - z(1 + D(z))} - 1 \right).$$

Use the implicit function scheme to derive an asymptotic formula for $[z^n]D(z)$.