## Analytic combinatorics
SS 2020

# Exercise sheet 5

Exercises for the exercise session on 8 June 2020

*Note.* The tree dissymmetry theorem (Theorem 6.3.2) is true for any class of unrooted trees, not only for the class of *all* trees. (Why?) In contrast, Corollaries 6.3.4 and 6.3.5 are *not* satisfied by all classes of unrooted trees.

**Problem 5.1.** Denote by $T(z)$ the exponential generating function of the class of labelled (unrooted) trees in which all vertices have degree 1 or 3. (Observe that every such tree has even order.)

(a) Apply the dissymmetry theorem to derive an expression of $T(z)$ in terms of $T^{\circ-\circ}(z)$.

(b) Determine the dominant singularities of $T^{\circ-\circ}(z)$ and singular expansions near those singularities.

(c) Use (a) and (b) to determine an asymptotic formula for $[z^n]T(z)$ for $n$ even.

*Hint.* For (a) and (b), you will in particular need to express $\mathcal{T}^{\circ}$ and $\mathcal{T}^{\circ-\circ}$, respectively, in terms of $\mathcal{T}^{\circ-\circ}$ and basic constructions. To do this, consider what is left from the tree if you delete the marked vertex or edge. How can we modify the remaining components so as to result in an element of $\mathcal{T}^{\circ-\circ}$?

**Problem 5.2.** Let $T(z)$ be as in Problem 5.1.

(a) Apply the dissymmetry theorem to derive an expression of $T(z)$ in terms of $T^{\circ}(z)$.

(b) Determine the dominant singularities of $T^{\circ}(z)$ and singular expansions near those singularities.

(c) Use (a) and (b) to determine an asymptotic formula for $[z^n]T(z)$ for $n$ even.

**Problem 5.3.** Let $P(z)$ be the EGF of permutations $\sigma$ with $\sigma^3 = \mathrm{id}$. Prove that $P(z)$ satisfies the conditions of Theorem 7.1.2. Approximate the solution $s_0(n)$ of the saddle-point equation up to a multiplicative error $(1 + o(1/n))$. In other words, find a function $f(n)$ so that

$$s_0 = f(n) + o\left(\frac{f(n)}{n}\right).$$

Use this to derive an upper bound for $[z^n]P(z)$.

*Hint.* In order to find $f(n)$, start by choosing $s$ so that the largest term on the left-hand side of the saddle-point equation equals $n$. This will create a smaller-order term on the left-hand side that is still $\omega(1)$. Try to change $s$ a little bit so that this term vanishes. Repeat this step until your left-hand side equals $n + 1 + o(1)$.

**Problem 5.4.** Suppose that $G(z), H(z)$ are both Hayman admissible with the same radius of convergence $\rho$. Prove that the function $F(z) := G(z)H(z)$ satisfies the capture condition of Hayman admissibility and that there exist $\rho_0 \in (0, \rho)$ and functions

$$\theta_0^-, \theta_0^+ \colon (\rho_0, \rho) \to (0, \pi)$$

such that $F(z)$ satisfies the locality condition for $|\theta| \leq \theta_0^-$ and the decay condition for $|\theta| > \theta_0^+$. (Meta question: Where does the difficulty lie in finding a single function $\theta_0$ that works for both conditions?)