Exercise sheet 6

Exercises for the exercise session on 17 June 2020

Problem 6.1. Let

$$F(z) = \frac{z}{1-z} \qquad \text{und} \qquad G(z) = \exp\left(\frac{z}{1-z}\right).$$

Prove that $F(z)$ is *not* Hayman admissible. Furthermore, prove that $G(z)$ satisfies the capture condition and determine which functions θ_0 satisfy the necessary condition $b(s)^{-1/2} \ll \theta_0 \ll c(s)^{-1/3}$ for the locality condition and the decay condition.
Under the assumption that $G(z)$ is indeed Hayman admissible, determine an asymptotic expression for the coefficients $[z^n]G(z)$.

Problem 6.2. Denote by $F(z)$ the exponential generating function of permutations consisting only of cycles of length 2 and 3. Prove that the unique positive solution s_0 of the equation

$$s_0 \frac{F'(s_0)}{F(s_0)} = n$$

satisfies

$$s_0 = n^{1/3} - \frac{1}{3} + \frac{1}{9} n^{-1/3} - \frac{2}{81} n^{-2/3} + o(n^{-2/3}).$$

Determine an asymptotic formula for $[z^n]F(z)$ using saddle-point asymptotics.

Comment: Feel free to use computer assistance for the more tedious calculations (in particular for the asymptotics of s_0^n).

Problem 6.3. Denote by $P(z)$ the ordinary generating function of plane rooted trees, and let $F(z) := e^{P(z)}$. Use saddle-point estimates of large powers to determine asymptotic formulae for $[z^n]P(z)$ and $[z^n]F(z)$.

Hint: First apply Lagrange inversion to express $[z^n]P(z)$ and $[z^n]F(z)$ by an expression to which saddle-point estimates of large powers can be applied.

Problem 6.4. Denote by $S(z)$ the ordinary generating function of bracketings (see also Problem 1.2). Apply the standard Lagrangean framework to derive an asymptotic expression for $[z^n]S(z)$.