Advanced Topics in Random Graphs
Exercise Sheet 2

Question 1. Let G be a graph on n vertices with $\delta(G) \geq k$ and let $p_1 = \frac{1+\frac{\varepsilon}{2}}{k}$ with ε a sufficiently small positive constant. Let $\gamma > 0$, show if $\delta > 0$ is sufficiently small then whp every set $B \subseteq V(G)$ of size $|B| \geq \gamma n$ and $e_G(B) \leq \delta |B| k$ has $e_{G_p}(B) \leq \frac{|B|}{2}$.

Question 2. Let G be as in question 1 and let $p = \frac{1+\varepsilon}{k}$. Assume that for each $v \in V(G)$ the probability that v lies in a component of G_p of size at least $\frac{\varepsilon k}{2}$ is at least $\frac{\varepsilon}{6}$. Show that there exists a constant $c > 0$ such that expected number of excess edges of G_p is at least cn.

(* Show that the property you assumed holds)

Question 3. Let G be as in Question 1. Suppose that we explore the component structure of G_p using the DFS process and let G' be the unqueried edges. Show that whp there is some component T of the DFS tree such that the average degree of $G'[T]$ is $\Theta(k)$. Deduce that for a sufficiently small constant $\delta > 0$ there are $\Theta(k^2)$ edges in $G'[T]$ whose distance in T is at least δk.

Deduce that whp G_p contains a cycle of length at least δk.

Question 4. Let $0 \leq p \leq 1/2$. Show that

$$\sum_{i \leq pn} \binom{n}{i} \leq 2^{h(p)n}.$$

where $h(p)$ is the entropy of a Bernoulli random variable with success probability p.

(*) Show further, that if $\mathcal{C} \subseteq 2^{[n]}$ and $0 \leq p \leq 1$ is the average of $|C|/n$ over $C \in \mathcal{C}$ then $|\mathcal{C}| \leq 2^{h(p)n}$.

Question 5. Suppose we have n coins, some of which weigh a grams, and some of which are counterfeit and weigh $b < a$ grams. We have a set of scales on which we can weigh any subset of the coins.

Suppose we have to choose a sequence of subsets to weigh beforehand, so that we cannot use the partial information to inform our future choices, what is the smallest number needed to find all the counterfeit coins?

(Hint: Let $\{A_1, \ldots, A_m\}$ be the subsets chosen, X be a random subset of $[n]$ and $X_i = |A_i \cap X|$ for each i)