Advanced Topics in Random Graphs
Exercise Sheet 4

Question 1. Given a graph $G = (V, E)$ with $|V| = 2n$ consider the permanent of it's adjacency matrix A. Show that every permutation σ of $[2n]$ with non-zero contribution to $\text{perm}(A)$ corresponds to a cover of the vertices of G by cycles and isolated edges, which we call a cycle cover.

Conversely, given a pair of perfect matchings M_1 and M_2 of G show that $M_1 \cup M_2$ is a subgraph of G covering the vertices with even length cycles and isolated edges, which we call an even cycle cover.

Using the above show that $|\Phi(G) \times \Phi(G)| \leq \text{perm}(A)$ and show that

$$\phi(G) \leq \prod_{v \in V} (d(v)!)^{\frac{1}{2d(v)}} .$$

Question 2. Let $X = (X_1, \ldots, X_n)$ be a discrete random variable. Show that there are non-negative constants $h_1, \ldots, h_n$ such that $H(X) = \sum_i h_i$ and

$$\sum_{i \in I} h_i \leq H(X_I)$$

for every $I \subseteq I$.

(Hint : Consider the proof of the Bollobás-Thomason Box Theorem in the notes).

Question 3. Let $\mathcal{G}$ be a set of graphs on $[n]$ such that for every pair of graphs $G_i, G_j \in \mathcal{G}$ there is an edge in their intersection $G_i \cap G_j$. How large can $\mathcal{G}$ be?

Suppose instead that we insist that there is a triangle in each intersection. Give an explicit example of a family of size $2^{\binom{n}{2}-3}$ with this property.

Let us presume for ease of presentation that n is even. We consider each G_i as a subset of the set $U = \binom{n}{2}$, and for each equipartition $[n] = A \cup B$ such that $|A| = |B|$, let $U(A, B)$ be the set of edges which lie entirely in A or B.

Show that the trace of $\mathcal{G}$ on any $U(A, B)$ is an intersecting family. Hence by considering the trace of $\mathcal{G}$ over the family $\mathcal{F} = \{U(A, B) : (A, B) \text{ an equipartition}\}$ show that

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-2}.$$

Question 4. Suppose $\mathcal{G}$ is set of graphs on $[n]$ such that for every pair of graphs $G_i, G_j \in \mathcal{G}$ the intersection $G_i \cap G_j$ contains no isolated vertices. Prove an upper bound for $|\mathcal{G}|$ and show that it is tight.