Advanced Topics in Random Graphs
Exercise Sheet 5

Question 1. Prove Theorems 5.6 and 5.8.

Question 2. Let H be a graph and let $\mathcal{H}$ be the $e(H)$-uniform hypergraph on $E(K_n)$ encoding copies of H. Show that there is some c such that $\mathcal{H}$ satisfies the container lemma for $\tau = n^{-\frac{1}{m_2(H)}}$ where $m_2(H)$ is the maximum subgraph density of H.

Give a container theorem for H-free graphs.

Question 3. Give a supersaturation version of the Erdős-Stone-Simonivits theorem and prove Theorem 5.14.

Question 4. Show that there exist constants $\delta > 0$ and $k_0 \in \mathbb{N}$ such that the following holds for every $k \geq k_0$ and $n \in \mathbb{N}$. Given a graph G on n vertices with $kn^{\frac{3}{2}}$ many edges, there exists a collection $\mathcal{H}$ of at least $\delta^3 k^4 n^2$ copies of C_4 in G such that:

a) Each edge belongs to at most $\delta^2 k^3 \sqrt{n}$ members of $\mathcal{H}$;

b) Each pair of edges is contained in at most $\delta k \sqrt{n}$ members of $\mathcal{H}$.

(Hint : You can find a large family of C_4s by double counting paths of length 2. Try to build a suitable collection one by one by keeping track of the number of C_4s we can add which don't violate a) or b).)

Question 5. By repeatedly applying the container lemma to the hypergraphs given by the previous question, show that there exists constants k_0 and $C > 0$ such for all $k_0 \leq k \leq \frac{n^{\frac{1}{6}}}{\log n}$ there exists a collection $\mathcal{G}(n,k)$ of at most

$$\exp\left(O\left(\frac{\log k}{k} n^{\frac{3}{2}}\right)\right)$$

many graphs on n vertices such that $e(G) \leq kn^{\frac{3}{2}}$ for each $G \in \mathcal{G}(n,k)$ and every C_4-free graph on n vertices is contained in some $G \in \mathcal{G}(n,k)$.

Deduce that there are at most $2^{O\left(n^{\frac{3}{2}}\right)}$ many C_4-free graphs on n vertices.