Discrete and algebraic structures

Winter term 2019/20

Exercise sheet 1

Exercises for the exercise session on 10/10/2019

All vector spaces on this sheet are finite-dimensional.

Problem 1.1. Let U, V, W be vector spaces over a field K. Prove that there is a canonical isomorphism between the vector space of bilinear maps $\varphi\colon U \times V \to W$ and $\mathrm{Hom}(U, \mathrm{Hom}(V, W))$.

Problem 1.2. Let $\varphi\colon V_1 \times \cdots \times V_m \to P$ be a multilinear map and suppose that $\psi\colon V_1 \times \cdots \times V_m \to W$ is a tensor map.

(a) Prove that φ is a tensor map if and only there is a linear map $T\colon P \to W$ with $\psi = T \circ \varphi$.

(b) Suppose that φ is a tensor map and show that the map T from a) is *unique* if and only if $\langle \mathrm{im}\varphi \rangle = P$ and that it can be chosen to be a bijection if and only if $\dim(P) = \dim(W)$.

Problem 1.3. Let U, V, W be vector spaces over a field K and let $u_1, \ldots, u_k \in U$, $v_1, \ldots, v_k \in V$, and $w_1, \ldots, w_k \in W$.

(a) Prove that if $\sum_{i=1}^{k}(u_i \otimes v_i)$ has rank k, then both sets $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are linearly independent.

(b) Suppose that $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are linearly independent and $w_i \neq 0$ for all $i = 1, \ldots, k$. Show that

$$\sum_{i=1}^{k}(u_i \otimes v_i \otimes w_i) \in U \otimes V \otimes W$$

has rank k.

Problem 1.4. Let $V_1, \ldots, V_m$ be vector spaces over a field K and suppose that, for each $i = 1, \ldots, m$, there are $v_{i,1}, \ldots, v_{i,k} \in V_i$ such that

$$\sum_{j=1}^{k} v_{1,j} \otimes \cdots \otimes v_{m,j} = 0.$$

Prove that if $v_{1,1}, \ldots, v_{1,k}$ are linearly independent, then for each $j = 1, \ldots, k$, at least one of $v_{2,j}, \ldots, v_{m,j}$ is zero.

Problem 1.5. Let $V_1, \ldots, V_m, W$ be vector spaces over a field K and suppose that there exists a surjective multilinear map $\varphi\colon V_1 \times \cdots \times V_m \to W$. Prove that there exists a subspace U of $V_1 \otimes \cdots \otimes V_m$ such that the quotient space $(V_1 \otimes \cdots \otimes V_m)/U$ is isomorphic to W and each of its equivalence classes contains a decomposable tensor.