Discrete and algebraic structures

Winter term 2019/20

Exercise sheet 4

Exercises for the exercise session on 7/11/2019

Problem 4.1. Let R be a ring with unit and let M be an R-module. Prove that for any positive integer n, a map $f\colon R^n \to M$ is an R-morphism if and only if there exist $m_1, \dots, m_n \in M$ such that

$$f\big((r_1, \dots, r_n)\big) = r_1 m_1 + \cdots + r_n m_m \,.$$

Problem 4.2. Suppose that in the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

of R-morphisms, the upper row is exact, while the lower row is semi-exact. Show that there exists a unique R-morphism $\gamma\colon C \to C'$ for which the completed diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\exists!\gamma} & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

is commutative.

Problem 4.3. If

$$0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence of R-modules, then we say that (f, E, g) is an *extension of* A *by* B. We call two extensions $(f_1, E_1, g_1), (f_2, E_2, g_2)$ of A by B *equivalent* if there exists an R-morphism $h\colon E_1 \to E_2$ with $h \circ f_1 = f_2$ and $g_2 \circ h = g_1$.

Prove that for any R-modules A, B, there exists an extension of A by B and that if $(f_1, E_1, g_1), (f_2, E_2, g_2)$ are equivalent extensions, then any R-morphism $h\colon E_1 \to E_2$ witnessing this equivalence is an isomorphism. Deduce from this that there exist non-equivalent extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_4$.

Problem 4.4. Let M be an R-module.

(a) Prove that if M is finitely generated, then so is every quotient module of M, and that if there is a finitely generated submodule N of M such that M/N is finitely generated, then M is also finitely generated.

(b) If M is finitely generated, is every submodule of M finitely generated as well?

Problem 4.5. Let R be a commutative ring with unit. We call an R-module M *cyclic* if there is an element $x \in M$ such that $M = \langle x \rangle$.

Suppose that $M = \langle x \rangle$ is cyclic. Prove that $\mathrm{Ann}_R\{x\} = \mathrm{Ann}_R M$ (the annihilator Ann_R was defined in Problem 3.4) and that $M \cong R/\mathrm{Ann}_R M$ (as R-modules). Deduce that two cyclic R-modules are isomorphic if and only if they have the same annihilator.

Problem 4.6. Consider the $\mathbb{Z}$-module $M = \mathbb{R}/\mathbb{Z}$. Prove that

$$N := \{m \in M \mid \exists z \in \mathbb{Z} \setminus \{0\} \colon zm = 0_M\}$$

is a submodule of M and that $M/N \cong \mathbb{R}/\mathbb{Q}$.