Discrete and algebraic structures
Winter term 2019/20

TU Graz

Exercise sheet 5

Exercises for the exercise session on 21/11/2019

Problem 5.1. Let $f, g, h\colon \mathbb{N} \to \mathbb{R}^+$. Prove or disprove the following claims.

(i) $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$;

(ii) $f(n) = o(g(n))$ if and only if $\frac{1}{f(n)} = \omega\left(\frac{1}{g(n)}\right)$;

(iii) $f(n) = O(g(n) + h(n))$ if and only if $f(n) = O(g(n))$ or $f(n) = O(h(n))$;

(iv) If $f(n) = o(g(n))$ and $g(n) = O(h(n))$, then $f(n) = o(h(n))$.

Problem 5.2. Let $f, g, h\colon \mathbb{N} \to \mathbb{R} \setminus \{0\}$ be given such that

$$f(n) = O(h(n)), \quad g(n) = O(h(n)), \quad \text{and} \quad h(n) = o(1).$$

Prove that

$$f(n) + g(n) = O(h(n)), \quad f(n) \cdot g(n) = o(h(n)), \quad \text{and} \quad \frac{1}{1 + f(n)} = 1 + O(h(n)).$$

Problem 5.3. Use the Problem 5.2 to prove that

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Problem 5.4. Let $f\colon \mathbb{N} \to \mathbb{R}$ be such that $f(n) = o(1)$. Show that

$$1 + f(n) = \exp\left(f(n) - \frac{(f(n))^2}{2} + O((f(n))^3)\right),$$
$$\exp(f(n)) = 1 + f(n) + O((f(n))^2).$$

Problem 5.5. Use Problem 5.4 to show that

$$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + O\left(\frac{1}{n^2}\right).$$

Problem 5.6. Let $A(z) = \sum_{n \geq 0} a_n z^n$ be a formal power series.

(a) Prove that $A(z)$ has a reciprocal if and only if $a_0 \neq 0$. Also prove that the reciprocal is unique if it exists.

(b) Suppose that all a_n are integers and prove that the infinite sum

$$B(z) := 1 + A(z) + A(z)^2 + \cdots$$

is a well-defined formal power series (i.e. it can be written as $B(z) = \sum_{n \geq 0} b_n z^n$) if and only if $a_0 = 0$. Furthermore, show that $B(z)$ is the reciprocal of $1 - A(z)$.