Discrete and algebraic structures
Winter term 2019/20

Exercise sheet 6
Exercises for the exercise session on 28/11/2019

Problem 6.1. Using only the identity

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n$$

and the basic operations for power series (sum, product, differentiation, integration), determine the complex functions that have the following power series.

$$\text{(i)} \sum_{n\geq 0} \frac{n}{n+1} z^n \qquad \text{(ii)} \sum_{n\geq 0} \left(\sum_{k=1}^{n} \frac{1}{k} \right) z^n \qquad \text{(iii)} \sum_{n\geq 2} n^2 z^n$$

Problem 6.2. Let $\mathcal{D}$ and $\mathcal{I}$ be the class of *derangements* (permutations without fixed-points) and *involutions* (self-inverse permutations), respectively.

(a) Use the symbolic method to determine the exponential generating function $D(z)$ of $\mathcal{D}$ and derive a sum formula for $[z^n]D(z)$.

(b) Use the symbolic method to determine the exponential generating function $I(z)$ of $\mathcal{I}$ and derive a sum formula for $[z^n]I(z)$.

Problem 6.3. Suppose that the complex function $A(z)$ satisfies

$$z = \frac{A(z)}{1 - A(z)}$$

for all z in some open ball around 0. Determine the coefficients $[z^n]A(z)$ in two different ways: once via Lagrange Inversion and then a second time by deducing the explicit form of $A(z)$ from the above equation.

Problem 6.4. Denote by $T(z)$ the exponential generating function of rooted labelled trees and let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Determine $[z^n]\exp(\alpha T(z))$ for all integers $n \geq 0$ and deduce from the result that

$$(\alpha + \beta)(n + \alpha + \beta)^{n-1} = \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (k + \alpha)^{k-1} (n - k + \beta)^{n-k-1}$$

for all such α, β, n with $\alpha + \beta \neq 0$.

Problem 6.5. Consider an unlabelled tree with a root vertex r. Every vertex $v \neq r$ has a unique neighbour w that is closer to r than v. In this case, we say that v is a *child* of w. If $u \neq v$ is also child of w, we call u, v *siblings*.
For an integer $k \geq 1$, let C be a set of k colours und let $\mathcal{T}_k$ be class of unlabelled rooted trees in which every every vertex apart from the root is coloured with a colour in C (i.e. r has no colour). Determine the number of

(a) trees in $\mathcal{T}_k$ with $n \geq 1$ vertices in which there are no two siblings with the same colour;

(b) trees in $\mathcal{T}_k$ with $n \geq m + 1$ vertices in which the root has precisely m children (where m is a fixed integer with $1 \leq m \leq k$) and there are no two siblings with the same colour.