Discrete and algebraic structures
Winter term 2019/20

Exercise sheet 7
Exercises for the exercise session on 5/12/2019

Problem 7.1. For $n \geq 1$, the *triangular grid T_n of size n* consists of all integer points (x, y) with $x, y \geq 0$ and $x + y \leq n$, together with directed edges of the type $(x, y) \to (x + 1, y)$, $(x, y) \to (x, y + 1)$, and $(x, y + 1) \to (x + 1, y)$ whenever $x + y \leq n - 1$.

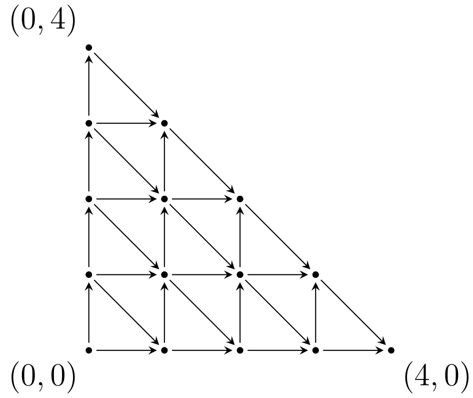

Figure 1: The triangular grid of size 4.

A path in T_n is called *valid* if it follows the directions of the edges. Denote by p_n the number of valid paths in T_n from $(0, 0)$ to $(n, 0)$ and let $P(z)$ be their ordinary generating function.

(a) Use the symbolic method to show that $P(z)$ satisfies the equation

$$P(z) = \frac{1}{1 - 2z - zP(z)} - 1$$

and deduce that

$$P(z) = \frac{1 - 3z - \sqrt{1 - 6z + z^2}}{2z}.$$

(b) Use singularity analysis to derive the asymptotic size of $p_n = [z^n]P(z)$ as

$$p_n = an^b c^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

for some constants a, b, c.

Hint. Formally, $P(z)$ is not defined at $z = 0$. Start by showing that $\lim_{z \to 0} P(z)$ exists. (Then complex analysis tells us that P can be extended to a function that is analytic at $z = 0$).

Problem 7.2. Let $c \in \mathbb{C}$, $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and an analytic function $f : \mathbb{C} \to \mathbb{C}$ be given. Determine the asymptotic value of $[z^{2n}]f(z^2)(1-z^2)^{-\alpha}$

(i) directly, i.e. by applying singularity analysis to $f(z^2)(1-z^2)^{-\alpha}$;

(ii) by applying singularity analysis to the function $f(z)(1-z)^{-\alpha}$ and using that $[z^{2n}]f(z^2)(1-z^2)^{-\alpha} = [z^n]f(z)(1-z)^{-\alpha}$.

Problem 7.3. Prove Hall's theorem that a bipartite graph with bipartition $\{A, B\}$ has a matching covering A if and only if

$$|N(S)| \geq |S| \quad \text{for every } S \subseteq A.$$

Also give an example that shows that Hall's theorem fails for graphs with infinitely many vertices.

Problem 7.4. Let k, n be positive integers and let X be a set of size kn. Prove that for any two partitions

$$X = \biguplus_{i=1}^{n} U_i \qquad \text{and} \qquad X = \biguplus_{i=1}^{n} V_i \qquad \text{with } |U_i| = |V_i| = k \text{ for all } i$$

there exists a common set of representatives $Y \subseteq X$ (i.e. $|U_i \cap Y| = |V_i \cap Y| = 1$ for all i). Show that this is not true if we start with three partitions.

Problem 7.5. Let d, n be positive integers. Prove that every connected graph G on n vertices with $\delta(G) \geq d$ contains a path of length

$$k := \min\{2d, n-1\}$$

and, if $d \geq 2$, a cycle of length at least

$$\ell := \min\{d+1, n\}.$$

Show that this is best possible in the sense that for every choice of d, there are infinitely many values of n for which there exists a graph G on n vertices with minimum degree at least d such that G neither contains a path of length $k+1$ nor a cycle of length at least $\ell + 1$.

Hint. For the first part, start by considering some longest path P in G. Where are the neighbours of the first and last vertices of P? If P has shorter length than k, try to build from P a cycle with the same vertex set as P (what kind of configuration would enable us to do this, preferably by only changing few edges?), which can then in turn be extended to a path longer than P.