Exercise sheet 8

Exercises for the exercise session on 12/12/2019

Problem 8.1. Let $G = (V, E)$ be a plane graph and suppose that there exists a number $r \geq 3$ such that every face of G has at least r edges on its boundary. Prove that

$$|E| \leq \frac{r}{r-2}(|V| - 2)$$

and deduce from this that neither K_5 nor $K_{3,3}$ are planar.

Problem 8.2. A graph G is called *outerplanar* if it is planar in a way such that all vertices of G lie on the boundary of the outer face. Prove that the following statements are equivalent.

(i) G is outerplanar;

(ii) G contains neither K_4 nor $K_{2,3}$ as a minor;

(iii) G contains neither K_4 nor $K_{2,3}$ as a topological minor.

Problem 8.3. Let $G = (V, E)$ be a graph. Prove that the following statements are equivalent.

(i) G is a tree;

(ii) G is connected and $|E| = |V| - 1$;

(iii) G is maximally acyclic, i.e. G is acyclic, but adding an edge between any to non-adjacent vertices creates a cycle.

Problem 8.4. For $n \geq, 3$, let T be a tree on the vertex set $\{1, \dots, n\}$ and let $s = (s_1, \dots, s_{n-2})$ be its Prüfer code. Can we characterise the trees for which $s_1, \dots, s_{n-2}$ are pairwise distinct? What about the case $s_1 = \dots = s_{n-2}$?
Construct a bijection similar to Prüfer codes that can be used to count *rooted* trees on $\{1, \dots, n\}$. (It is not necessary to re-prove statements that are already known für Prüfer codes; just point out the differences to your construction.)

Problem 8.5. Let $G = (V, E)$ be a graph.

(a) Suppose that G contains a matching M consisting of m edges. Prove that there exists a set $S \subset V$ such that there are at least

$$\frac{|E| + m}{2}$$

edges between S and $V \setminus S$.

(b) Suppose that $|E| \leq c^2$ for some fixed $c \in \mathbb{N}$. Prove that it is possible to assign a colour to each vertex of G so that at most $2c$ different colours are used in total and for each edge, its end vertices have different colours.

Note/Hint. Both parts can be proved by determining the expectation of a suitable random variable. For (a), choosing S randomly among *all* subsets of V would only suffice if we aimed for $\frac{|E|}{2}$ edges. For (b), first consider a random colouring using only c colours.