Probabilistic method in combinatorics and algorithmics

WS 2019/20

Exercise sheet 1

Exercises for the exercise session on 9 October 2019

Definition. A *k-uniform hypergraph* is a pair $H = (V, E)$ with vertex set V and hyperedge set E, where every hyperedge is a subset of V containing exactly k elements.

Problem 1.1. Prove that if a k-uniform hypergraph H has fewer than 2^{k-1} edges, then one can colour the vertices of H by two colours so that there is no edge whose vertices are all coloured by a single colour.

Problem 1.2. Let $k \geq 4$. Prove that if a k-uniform hypergraph H has fewer than $4^{k-1}/3^k$ edges, then one can colour the vertices of H by four colours so that all four colours are represented in every edge.

Definition. A *tournament* is an orientation of a complete graph, i.e. for every pair of distinct vertices v, w, exactly one of the directed edges (v, w) and (w, v) is present. A *Hamiltonian path* in a tournament is a directed path passing through all vertices.

Problem 1.3. Prove that there exists a tournament on n vertices that has at least $n!\, 2^{-n+1}$ Hamiltonian paths.

Problem 1.4. Let G be a bipartite graph with n vertices and suppose that each vertex v has a list $S(v)$ of colours. Prove that if $|S(v)| > \log_2 n$ for each v, then we can colour every vertex with a colour from its list so that no two adjacent vertices have the same colour.
Hint. Partition the set $\bigcup_v S(v)$ into two random sets.

Problem 1.5. Prove that given any n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with $||\mathbf{v}_i|| = 1$ for all $1 \leq i \leq n$, there exist n numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{+1, -1\}$ such that

$$\left|\left|\sum_{i=1}^{n} \epsilon_i \mathbf{v}_i\right|\right| \leq \sqrt{n}.$$