Exercise sheet 2

Exercises for the exercise session on 23 October 2019

Problem 2.1. Prove that for $k, n \in \mathbb{N}$ satisfying

$$\binom{n}{k}\left(1-\left(\frac{1}{2}\right)^k\right)^{n-k} < 1,$$

there is a tournament on n vertices with the property that for every set of k players, there is some player who beats them all.

Problem 2.2. Suppose an unbiased coin is tossed n times. For $k \leq n$, let A_k denote the event that out of these n tosses, there are k consecutive ones with the same outcome (i.e. k consecutive 'heads' or k consecutive 'tails'). Let $\varepsilon > 0$. Prove that

(a) $\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} 0$ if $k \geq (1+\varepsilon)\log_2 n$;

(b) $\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} 1$ if $k \leq \log_2 n - (1+\varepsilon)\log_2\log_2 n$.

Problem 2.3. Call an edge in a graph *isolated* if both its end vertices lie in no other edge. Denote by X the number of isolated edges in $G(n, p)$.

(a) Determine $\mathbb{E}[X]$ and prove that for every given $\varepsilon > 0$,

$$\mathbb{E}[X] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p \geq n^{\varepsilon-1}, \\ \infty & \text{if } p \leq (1-\varepsilon)\frac{\ln n}{2n}, \text{ but } p = \omega\left(\frac{1}{n^2}\right). \end{cases}$$

Hint. For $\mathbb{E}[X] \to \infty$, it might help to split the interval for p into two parts.

(b) Prove that

$$\mathbb{P}(X \geq 1) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } \mathbb{E}[X] \to 0, \\ 1 & \text{if } \mathbb{E}[X] \to \infty. \end{cases}$$

Problem 2.4. Let $n \geq k \geq 1$ be integers.

(a) Prove that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{en}{k}\right)^k.$$

(b) For any constant $\alpha \in (0, 1)$, show

$$\binom{n}{\alpha n} = 2^{H(\alpha)n + O(\log_2 n)},$$

where $H\colon (0, 1) \to \mathbb{R}$ is defined by

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x).$$

Prove that the same formula is still true if α is not constant, but satisfies

$$\alpha = \omega\left(\frac{1}{n}\right) \qquad \text{and} \qquad 1-\alpha = \omega\left(\frac{1}{n}\right).$$

Problem 2.5. (a) Let $x \in \mathbb{R}$ be given. Prove that $1 + x \leq \exp(x)$. Furthermore, prove that $1 + x \geq \exp\left(x - \frac{x^2}{2}\right)$ is true if and only if $x \geq 0$.

(b) Let integers $n \geq k \geq 1$ be given. Use (a) to show that the falling factorial $(n)_k := \frac{n!}{(n-k)!}$ satisfies

$$n^k \exp\left(-\frac{k(k-1)}{2(n-k+1)}\right) \leq (n)_k \leq n^k \exp\left(-\frac{k(k-1)}{2n}\right).$$