# Exercise sheet 3

Exercises for the exercise session on 5 November 2019

**Problem 3.1.** Let $n \geq k \geq 1$ be integers.

(a) Prove that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{en}{k}\right)^k.$$

(b) For any constant $\alpha \in (0,1)$, show

$$\binom{n}{\alpha n} = 2^{H(\alpha)n + O(\log_2 n)},$$

where $H\colon (0,1) \to \mathbb{R}$ is defined by

$$H(x) = -x \log_2 x - (1-x)\log_2(1-x).$$

Prove that the same formula is still true if $\alpha$ is not constant, but satisfies

$$\alpha = \omega\left(\frac{1}{n}\right) \qquad \text{and} \qquad 1 - \alpha = \omega\left(\frac{1}{n}\right).$$

**Problem 3.2.** (a) Let $x \in \mathbb{R}$ be given. Prove that $1 + x \leq \exp(x)$. Furthermore, prove that $1 + x \geq \exp\left(x - \frac{x^2}{2}\right)$ is true if and only if $x \geq 0$.

(b) Let integers $n \geq k \geq 1$ be given. Use ((a)) to show that the falling factorial $(n)_k := \frac{n!}{(n-k)!}$ satisfies

$$n^k \exp\left(-\frac{k(k-1)}{2(n-k+1)}\right) \leq (n)_k \leq n^k \exp\left(-\frac{k(k-1)}{2n}\right).$$

**Problem 3.3.** Let $A_1, A_2, A_3$ be events in a probability space such that

- $A_1$ and $A_2$ are independent;
- $A_2$ and $A_3$ are independent;
- $A_1 \cap A_2$ and $A_3$ are independent.

Prove that $A_1$ and $A_2 \cap A_3$ are independent as well. Deduce that one may assume that the dependency graph $D$ in the Lovász Local Lemma has the following properties.

(i) Every vertex $i$ of $D$ is either isolated (i.e. there are no edges involving $i$) or has at least one outgoing edge (i.e. an edge $(i, j)$).

(ii) Either $D$ is edgeless or contains a directed cycle (this cycle might consist of only two edges $(i, j)$ and $(j, i)$).

**Problem 3.4.** Let $n \geq 2$ be fixed. For the following probability spaces and sets of 'bad' events $A_1, \ldots, A_n$, prove that the Lovász Local Lemma *cannot* be applied (regardless of the choice of the dependency graph $D$).

(a) Let $\Omega$ be the set of all sequences $s_1, \ldots, s_n$ with $s_i \in \{1, \ldots, 6\}$ such that $s_1 + \cdots + s_n$ is even. Choose an element from $\Omega$ uniformly at random. (I.e. we are considering the outcome of throwing a fair die $n$ times, conditioned on the event that the sum of values is even.) Denote by $A_i$ the event that $s_i$ is odd.

(b) Let $\Omega$ be the set of all permutations of $[n]$. We choose a permutation $\sigma$ uniformly at random and denote, for each $i \in [n]$, by $A_i$ the event that $\sigma(i) = i$.

**Problem 3.5.** Let $A_1, \ldots, A_n$ be events in a probability space and let $D = ([n], E)$ be a directed graph. We say that $D$ is a *negative dependency graph* for $A_1, \ldots, A_n$ if the following holds.

For every $J \subseteq [n]$ and $i \in [n] \setminus J$, if $(i, j) \notin E$ for all $j \in J$ and $\mathbb{P}\left[\bigwedge_{j\in J} \overline{A_j}\right] > 0$, then $\mathbb{P}\left[A_i \,\middle|\, \bigwedge_{j\in J} \overline{A_j}\right] \leq \mathbb{P}(A_i)$.

Replacing 'dependency graph' by 'negative dependency graph' in the statement of the Lovász Local Lemma results in the so-called *Lopsided Lovász Local Lemma.*

(a) Sketch how the proof of the Lovász Local Lemma from the lecture generalises to a proof of the Lopsided Lovász Local Lemma.

(b) Follow the arguments sketched below to prove in that Problem 3.4(b), the edgeless graph on $[n]$ is a negative dependency graph.

It suffices to show (why?) that

$$\mathbb{P}\left[\bigwedge_{j\in J} \overline{A_j} \,\middle|\, A_i\right] \leq \mathbb{P}\left[\bigwedge_{j\in J} \overline{A_j}\right]$$

for all $J \subset [n]$ and $i \in [n] \setminus J$ such that $\mathbb{P}[A_i] > 0$. Show that this is equivalent to

$$n\left|A_i \wedge \bigwedge_{j\in J} \overline{A_j}\right| \leq \left|\bigwedge_{j\in J} \overline{A_j}\right|.$$

To prove this inequality, for each permutation $\sigma \in A_i \wedge \bigwedge_{j\in J} \overline{A_j}$, define $n$ permutations $\sigma_1, \ldots, \sigma_n \in \bigwedge_{j\in J} \overline{A_j}$ and prove that $\sigma_k \neq \tau_l$ as soon as $k \neq l$ or $\sigma \neq \tau$.