Exercise sheet 4

Exercises for the exercise session on 13 November 2019

Problem 4.1. We say that a hypergraph $H = (V, E)$ is *2-colourable* if there exists a colouring of V by two colours so that no edge in E is monochromatic.
Let $k \geq 2$ be given.

(a) Let H be a hypergraph in which every edge contains at least k vertices. Suppose that each edge of H intersects at most $d \geq 1$ other edges. Prove that H is 2-colourable if $e(d+1)2^{1-k} \leq 1$.

(b) Suppose that H is a hypergraph in which each edge has at least k vertices. For each edge f and each $j \geq k$, denote by $d_{f,j}$ the number of edges of size j that intersect f. Prove that if for each edge f of H

$$8 \sum_{j \geq k} \frac{d_{f,j}}{2^j} \leq 1,$$

then H is 2-colourable.

Problem 4.2. The *Ramsey number* $R(t)$ is defined as the smallest integer n such that any graph G on n vertices contains either a clique of order t or an independent set of order t.

(a) Prove that if $e\left(\binom{t}{2}\binom{n-2}{t-2} + 1\right) 2^{1-\binom{t}{2}} \leq 1$, then $R(t) > n$.

(b) Prove that when $t \to \infty$,

$$R(t) \geq \frac{\sqrt{2}}{e}(1 + o(1))\, t\, 2^{t/2}.$$

Problem 4.3. Let G be a graph and let $d \geq 1$. Suppose that for every vertex v, there exists a list $S(v)$ of precisely $\lceil 2ed \rceil$ 'admissible' colours such that no colour in $S(v)$ is admissible for more than d neighbours of v. Prove that there is a 'proper' colouring of G (i.e. no two adjacent vertices have the same colour) assigning to each vertex an admissible colour.
(*Hint.* The fewer vertices and colours play a role in the probability of a 'bad' event A, the simpler the expression for $\mathbb{P}[A]$ will be.)

Problem 4.4. Let D be a directed graph without loops (i.e. $E(D)$ is a subset of $\{(u, v) \mid u, v \in V(D) \wedge u \neq v\}$) in which each vertex has precisely δ^+ many outgoing edges and at most Δ^- many ingoing edges. Suppose that k is a positive integer satisfying

$$e(\delta^+\Delta^- + 1)\left(1 - \frac{1}{k}\right)^{\delta^+} < 1.$$

Prove that there exists a colouring $c\colon V(D) \to \{0, \ldots, k-1\}$ such that each vertex $v \in V(D)$ has an outgoing edge (v, w) with $c(w) \equiv c(v) + 1 (\mathrm{mod} k)$.
Derive from this that if each vertex of D has *at least* δ^+ outgoing and at most Δ^- ingoing edges, then D contains a directed cycle whose length is a multiple of k.

Problem 4.5. Let $p = p(n) \in (0,1)$ be given.

(a) For $t > 0$ and a fixed vertex v of $G(n,p)$, compare the bounds on $\mathbb{P}[|d(v) - \mathbb{E}[d(v)]| \geq t]$ provided by Chebyshev's inequality and by the Chernoff bounds 1 and 2. In each of the three cases, how large does t have to be in order to deduce that

$$\mathbb{P}[|d(v) - \mathbb{E}[d(v)]| \geq t] = o(1)?$$

(b) How large does t have to be if we want to prove that

$$\mathbb{P}[\exists v \in [n]\colon\ |d(v) - \mathbb{E}[d(v)]| \geq t] = o(1)?$$

(c) Are there functions $p(n)$ for which the minimum requirements for t in (b) from the two Chernoff bounds coincide?