Exercise sheet 5

Exercises for the exercise session on 26 November 2019

Problem 5.1. Define the set $S \subset \mathbb{N}$ by letting each number n be in S with probability $1/2$ independently.

(a) For $k, l \in \mathbb{N}$, we set

$$w_l(k) = \left\lceil \frac{\ln(kl2^{k-1})}{\ln 2} \right\rceil.$$

Denote by A_l the event that there is a $k \geq 2$ such that S contains an arithmetic progression of the form

$$k - b, k, k + b, \ldots, k + (w_l(k) - 2)b.$$

Prove that $\mathbb{P}[A_l] \leq 1/l$ and deduce from this that with probability 1, S does not contain an arithmetic progression of infinite length.

(b) Prove that

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{|S \cap [n]|}{n} = \frac{1}{2}\right] = 1.$$

To that end, for fixed $\varepsilon > 0$ and n, use the Chernoff bounds to find an upper bound for

$$\mathbb{P}\left[\left|\frac{|S \cap [n]|}{n} - \frac{1}{2}\right| \geq \varepsilon\right]$$

and apply a union bound to show that

$$\mathbb{P}\left[\exists n \geq n_0 \text{ with } \left|\frac{|S \cap [n]|}{n} - \frac{1}{2}\right| \geq \varepsilon\right] \overset{n_0\to\infty}{=} o(1).$$

Where does this strategy fail when we use Chebyshev's inequality instead of Chernoff bounds?

(*Note.* Szemerédi's Theorem states that each $A \subset \mathbb{N}$ with

$$\limsup_{n\to\infty} \frac{|A \cap [n]|}{n} > 0$$

contains infinitely many arithmetic progressions of length k for *every* k. Thus, Problem 5.1 shows that with probability 1, S contains arbitrarily long arithmetic progressions, but no arithmetic progression of infinite length.)

Problem 5.2. Suppose we place n balls in n bins, where each ball chooses its bin uniformly at random and independently from the other balls.

(a) Prove that for each $\varepsilon > 0$,

$$\mathbb{P}\left[\exists \text{ a bin with at least } \left(\frac{3}{2} + \varepsilon\right) \ln n \text{ balls}\right] = o(1).$$

(b) By how much can we decrease the value $\left(\frac{3}{2}+\varepsilon\right)\ln n$ in (a) so that we can still prove (by the same type of arguments as in (a)) that the probability is $o(1)$?

(c) If we have n^2 balls in total, for what $k = k(n)$ can we prove that

$$\mathbb{P}\left[\exists \text{ a bin with at least } k \text{ balls}\right] = o(1)?$$

Problem 5.3. Suppose that an urn contains one red ball and one blue ball. A ball is drawn from the urn uniformly at random. After that, the ball is put back into the urn and another ball of the same colour is added to the urn. This process is repeated n times. Denote by X_n the proportion of red balls in the urn after these n steps (i.e. number of red balls divided by total number of balls). Use Azuma's inequality to prove that

$$\mathbb{P}\left[\left|X_n - \frac{1}{2}\right| \geq \varepsilon\right] < 2\exp\left(-\frac{6\varepsilon^2}{2\pi^2 - 15}\right).$$

Problem 5.4. Let $S_1, \ldots, S_m$ be finite sets. Independently for each S_i, consider an arbitrary probability distribution $\mathbb{P}_i$. Let $\mathbb{P}$ be the probability distribution on

$$\Omega := \{(s_1, \ldots, s_m) \mid \forall 1 \leq i \leq m \colon s_i \in S_i\}$$

in which the coordinate s_i is chosen according to $\mathbb{P}_i$, independently from the other coordinates. Let $f \colon \Omega \to \mathbb{R}$ be a function. For every $\sigma = (s_1, \ldots, s_m) \in \Omega$, we choose $\tau = (t_1, \ldots, t_m) \in \Omega$ according to $\mathbb{P}$ and set, for $i = 0, \ldots, m$,

$$X_i(\sigma) := \mathbb{E}[f(\tau) \mid \forall 1 \leq j \leq i \colon s_j = t_j].$$

Then X_i is a random variable on Ω.

(a) Prove that $X_0, \ldots, X_m$ is a martingale. Deduce from this that in particular, the edge exposure martingale and the vertex exposure martingale are indeed martingales.

(b) Prove that if $|f(\sigma_1) - f(\sigma_2)| \leq 1$ holds for all $\sigma_1, \sigma_2 \in \Omega$ that differ in only one coordinate, then we have

$$|X_i(\sigma) - X_{i-1}(\sigma)| \leq 1$$

for every $\sigma \in \Omega$ and all $i = 1, \ldots, m$.

Problem 5.5. For an integer $n \geq 1$, let G be the graph with vertex set $V(G) = (\mathbb{Z}_7)^n$ and with $\{u, v\} \in E(G)$ if and only if u and v differ in only one coordinate. Suppose that $U \subset V(G)$ with $|U| = 7^{n-1}$ is given. For every $c > 0$, we define W_c to be the set of vertices of G with distance at least $(2+c)\sqrt{n}$ from U. Show that

$$|W_c| < 7^n e^{-\frac{c^2}{2}}.$$

(*Hint.* Define a martingale $X_0, \ldots, X_n$ for which X_0 is the average distance—taken over all vertices of G—from U and $X_n(v)$ is the distance of v from U. Apply Azuma's inequality twice: first to prove that X_0 is 'small' and then to deduce the desired upper bound for W_c.)