Exercise sheet 6

Exercises for the exercise session on 4 December 2019

Problem 6.1. Let $h = h(n)$ be a positive-valued function with $h(n) = \omega(1)$ (i.e. $h \to \infty$ as $n \to \infty$), but $h(n) = o(\ln n)$. Given

$$m = (\ln n - h) \cdot n$$

balls and n bins, place each ball into a bin chosen uniformly at random, independently for each ball. Prove that

$$\mathbb{P}[\exists \text{ empty bin}] \overset{n\to\infty}{\longrightarrow} 1.$$

Problem 6.2. Let $0 < p < \frac{1}{2}$ be a constant and let $X_1, X_2, \dots$ be i.i.d. random variables with

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } 1-p. \end{cases}$$

Set $Y_0 := 1$ and

$$Y_i := \left(\frac{1-p}{p}\right)^{X_1+\dots+X_i} \qquad \text{for } i \geq 1.$$

(a) Verify that $Y_0, \dots, Y_n$ is a martingale (for fixed n).

(b) Show that there exists a constant $0 < q < 1$ (depending on p) such that

$$\mathbb{P}\left[X_1 + \dots + X_n \geq 0\right] < q^n$$

for all n and prove that this implies

$$\lim_{k\to\infty} \mathbb{P}\left[\exists n \geq k \colon Y_n \geq 1\right] = 0.$$

(*Hint.* The existence of q can either be proved via Chernoff bounds or with the help of an 'exposure' martingale.)

Problem 6.3. Let n be a positive integer and let $p = p(n) \in (0, 1)$. Let X be the sum of n i.i.d. random variables $Y_1, \dots, Y_n$, which are 1 with probability p and 0 with probability $1-p$. Define a martingale $X_0, \dots, X_n$ that satisfies $X_0 = \mathbb{E}[X]$ and $X_n = X$. Compare the bound that Azuma's inequality gives for

$$\mathbb{P}[X > \mathbb{E}[X] + t]$$

with the bound from Chernoff's inequality. Which one is better? Does the answer depend on the choice of $p(n)$ and t?

Problem 6.4. A famous result about random graphs states that $G(n,p)$ has a perfect matching with probability $1-o(1)$ whenever n is even and $p \geq \frac{(1+\varepsilon)\ln(n)}{n}$. (Note that this is just large enough in order to guarantee that there are no isolated vertices.) Prove the following weaker result.
Denote by $m(G)$ the size of the largest matching in the graph G and write $\mu := \mathbb{E}[m(G(n,p))]$. Suppose that $p = p(n) \in [0,1]$ is such that $pn \to \infty$. Prove that for every $t > 0$,

$$\mathbb{P}\Big[m\big(G(n,p)\big) \leq \mu - t\sqrt{n-1}\Big] < \exp\left(-\frac{t^2}{2}\right)$$

and show that for every $\varepsilon > 0$,

$$\mu \geq (1-\varepsilon)\frac{pn}{2}$$

if n is large enough.

Problem 6.5. Let $k_0 = k_0(n) \in \mathbb{N}$ be such that

$$\binom{n}{k_0} 2^{-\binom{k_0}{2}} < 1 < \binom{n}{k_0 - 1} 2^{-\binom{k_0-1}{2}}$$

and let $k := k_0 - 4$. Use Janson's inequality to prove that

$$\mathbb{P}[G(n,1/2) \text{ contains } \textit{every} \text{ graph on } k \text{ vertices as an induced subgraph}] \xrightarrow{n\to\infty} 1.$$

Hint. Any ingredients of the applications of Janson's inequality from the lecture can be used without reproving them. Given a graph H on k vertices and a set $S \subset [n]$ of size k, there might be several ways how $G(n,1/2)$ can induce on S a graph isomorphic to H. The calculations will be easier if you consider just one fixed isomorphism.