Exercise sheet 3

Exercises for the exercise session on 13 April 2021

Problem 3.1. Show that the bivariate exponential generating function of permutations counted according to both the number of elements (marked by z) and the length of the cycle that contains the element 1 (marked by u) is

$$P(z,u) = \frac{1}{1-z} \log\left(\frac{1}{1-uz}\right).$$

Denote by X_n the length of the cycle that contains the element 1 in a random permutation which is chosen uniformly at random from all permutations of size n. Determine $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Problem 3.2. A *derangement* is a permutation without fixed-points. Determine the expected number of cycles in a random derangement which is chosen uniformly at random from all derangements of length n.

Problem 3.3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function and suppose that there is a point $z_0 \in \mathbb{C} \setminus \{0\}$ and a real number $R > |z_0|$ such that

- $f(z_0) = 0$;
- $f(z) \neq 0$ for all $z \neq z_0$ with $|z| < R$;
- $f'(z_0) \neq 0$.

(a) Prove that there exists a function $H(z)$ that is holomorphic on the open disc of radius R around the origin and satisfies

$$\frac{1}{f(z)} = \frac{1}{f'(z_0)(z-z_0)} + H(z).$$

(This in particular proves the missing part in Example 4.4.3 from the lecture.)

(b) Derive an asymptotic expression for $[z^n]\frac{1}{f(z)}$.

Problem 3.4. For $r, n \in \mathbb{N}$ and a surjection $\phi\colon [n] \to [r]$, we say that ϕ is *even* if all preimages $\phi^{-1}(i)$ for $i \in [r]$ have even size. Analogously, we call ϕ *odd* if each $\phi^{1}(i)$ has odd size. Denote by $\mathcal{E}_n^{(r)}$ and $\mathcal{O}_n^{(r)}$ the combinatorial classes of all even and of all odd surjections, respectively. Consider the exponential generating functions $E(z)$ and $O(z)$

$$\mathcal{E} = \sum_{n\geq 1}\sum_{r\geq 0} \mathcal{E}_n^{(r)} \qquad \text{and} \qquad \mathcal{O} = \sum_{n\geq 1}\sum_{r\geq 0} \mathcal{O}_n^{(r)},$$

respectively.

(a) Show that Theorem 4.4.2 from the lecture can be applied to $O(z)$ and determine an asymptotic expression for $[z^n]O(z)$.

(b) Show that Theorem 4.4.2 does *not* apply to $E(z)$.