Analytic combinatorics
Summer term 2021

Exercise sheet 4

Exercises for the exercise session on 26 April 2021

All asymptotic expressions on this sheet should be determined up to a multiplicative error of $1 + O(1/n)$. (If you can provide a stronger error term, even better, but not necessary.)

Problem 4.1. Let $B(z) = \sum_n B_n z^n$ and $S(z) = \sum_n S_n z^n$ denote the ordinary generating functions for the classes of binary strings with no consecutive 0's (note: the empty string is included in this class) and the class of bracketings, respectively. Recall from Problem 1.3 that

$$S(z) = \frac{1 + z - \sqrt{z^2 - 6z + 1}}{4}.$$

Derive a closed expression for $B(z)$ and use singularity analysis to find asymptotic expressions for B_n and S_n.

Problem 4.2. Consider the ordinary generating function $T(z) = \sum_n T_n z^n$ of triangulations and the exponential generating function $E(z) = \sum_n \frac{E_n}{n!} z^n$ of surjections in which every preimage has even size. We know from the lecture that

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

and from Problem 3.4 that

$$E(z) = \frac{1}{2 - \cosh(z)}.$$

Use singularity analysis to derive asymptotic expressions for T_n and E_n. Compare the result for T_n with the asymptotic expression obtained by applying Stirling's formula to the closed expression for T_n (see Example 1.3.9 in the lecture notes).

Problem 4.3. Let G_n denote the number of vertex-labelled 2-regular (simple) graphs on vertex set $[n]$, all whose components have even size.

(a) Derive a closed expression for the exponential generating function $G(z) = \sum_n \frac{G_n}{n!} z^n$.

(b) Use the transfer theorem for multiple singularities to derive an asymptotic expression for G_n.

Problem 4.4. For $r \geq 2$, let $\mathcal{C}_r$ be the class of Cayley trees in which every vertex has at most r children. Denote by $C_r(z)$ the exponential generating function of $\mathcal{C}_r$.

(a) Show that the singular inversion theorem can be applied to $C_r(z)$ for every r and determine an asymptotic formula for $[z^n]C_r(z)$ in the special case $r = 2$.

(b) Prove that the dominant singularity ρ_r of $C_r(z)$ converges to $\frac{1}{e}$ for $r \to \infty$.