Exercise sheet 1

Exercises for the exercise session on 13 October 2020

Definition. A *k-uniform hypergraph* is a pair $H = (V, E)$ with vertex set V and (hyper)edge set E, where every (hyper)edge is a subset of V containing exactly k elements.

Problem 1.1. Let $k \geq 2$. Prove that if a k-uniform hypergraph H has at most 2^{k-1} edges, then one can colour the vertices of H by two colours so that there is no *monochromatic* edge (that is, an edge whose vertices all have the same colour).

Problem 1.2. Let $k \geq 4$. Prove that if a k-uniform hypergraph H has at most

$$\frac{4^{k-1}}{3^k}$$

edges, then one can colour the vertices of H by four colours so that every edge is *rainbow* (that is, all four colours are represented among the vertices of the edge).

Definition. A *tournament* is an orientation of a complete graph, i.e. for every pair of distinct vertices v, w, exactly one of the directed edges (v, w) and (w, v) is present. A *Hamiltonian path* in a tournament is a directed path passing through all vertices.

Problem 1.3. Prove that for every $n \geq 3$, there exists a tournament on n vertices that has more than $n!\, 2^{-n+1}$ Hamiltonian paths.

Problem 1.4. Prove that for $k, n \in \mathbb{N}$ satisfying

$$\binom{n}{k}\left(1-\left(\frac{1}{2}\right)^k\right)^{n-k} < 1,$$

there is a tournament on n vertices with the property that for every set A of k vertices, there is some vertex v so that all edges between v and A are directed towards their end vertex in A.

Problem 1.5. Let G be a bipartite graph with n vertices and suppose that each vertex v has a list $S(v)$ of colours. Prove that if $|S(v)| > \log_2 n$ for each v, then we can colour every vertex with a colour from its list so that no two adjacent vertices have the same colour.
Hint. Partition the set $\bigcup_v S(v)$ into two random sets.

Problem 1.6. Let $n \in \mathbb{N}$ and let $\mathcal{F}$ be an inclusion-free family of subsets of $[n] := \{1, 2, \ldots, n\}$ (*inclusion-free* means that no element of $\mathcal{F}$ is a proper subset of another element). Choose a permutation σ of $[n]$ uniformly at random and define the random variable

$$X := |\{k \mid \{\sigma(1), \sigma(2), \ldots, \sigma(k)\} \in \mathcal{F}\}|.$$

Consider $\mathbb{E}[X]$ in order to prove that $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.