# Exercise sheet 2

Exercises for the exercise session on 22 October 2020

**Problem 2.1.**

(a) Let $x \in \mathbb{R}$. Prove that $1+x \leq \exp(x)$ and show that $1+x \geq \exp(x - \frac{x^2}{2})$ holds if $x \geq 0$.

(b) For any constant $\alpha \in (0,1)$, show

$$\binom{n}{\alpha n} = 2^{H(\alpha)n + O(\log_2 n)},$$

where $H\colon (0,1) \to \mathbb{R}$ is defined by

$$H(x) = -x \log_2 x - (1-x)\log_2(1-x).$$

Prove that the same formula is still true if $\alpha$ is not constant, but satisfies

$$\alpha = \omega\left(\frac{1}{n}\right) \qquad \text{and} \qquad 1-\alpha = \omega\left(\frac{1}{n}\right).$$

**Problem 2.2.** Let $k = k(n)$ be such that $k \to \infty$ as $n \to \infty$, but $k = o(n^{2/3})$. Use Stirling's formula

$$n! = \big(1+o(1)\big)\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n$$

and the Taylor expansion for $\ln(1+x)$ to prove that

$$\binom{n}{k} = \big(1+o(1)\big)\frac{1}{\sqrt{2\pi k}}\left(\frac{n}{k}\right)^k \cdot \begin{cases} \exp(k) & \text{if } k = o(n^{1/2}), \\ \exp\left(k - \frac{k^2}{2n}\right) & \text{if } k = \Omega(n^{1/2}). \end{cases}$$

What exponential term would be needed if $k = \Omega(n^{2/3})$, but $k = o(n^{3/4})$?

**Definition.** A property $\mathcal{P}$ of graphs is called *decreasing* if adding edges to a graph that does *not* have property $\mathcal{P}$ always results in a graph without property $\mathcal{P}$.

**Problem 2.3.** Prove that if $\mathcal{P}$ is a decreasing property of graphs, then

$$f\colon [0,1] \to [0,1], \quad f_{\mathcal{P}}(p) := \mathbb{P}[G(n,p) \text{ has property } \mathcal{P}]$$

is a decreasing function.

*Hint.* Given probabilities $p_1 < p_2$, try to find a third probability $p_0$ for which $G(n,p_0) \cup G(n,p_1)$ has the same probability distribution as $G(n,p_2)$.

**Problem 2.4.** Suppose an unbiased coin is tossed $n$ times. For $k \leq n$, let $A_k$ denote the event that out of these $n$ tosses, there are $k$ consecutive ones with the same outcome (i.e. $k$ consecutive 'heads' or $k$ consecutive 'tails'). Let $\varepsilon > 0$. Prove that

(a) $\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} 0$ if $k \geq (1+\varepsilon)\log_2 n$;

(b) $\mathbb{P}(A_k) \overset{n\to\infty}{\longrightarrow} 1$ if $k \leq \log_2 n - (1+\varepsilon)\log_2 \log_2 n$.

**Problem 2.5.** Call an edge in a graph *isolated* if both its end vertices lie in no other edge. Denote by $X$ the number of isolated edges in $G(n,p)$.

(a) Determine $\mathbb{E}[X]$ and prove that for every given $\varepsilon > 0$,

$$\mathbb{E}[X] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p \geq n^{\varepsilon-1}, \\ \infty & \text{if } p \leq (1-\varepsilon)\frac{\ln n}{2n}, \text{ but } p = \omega\left(\frac{1}{n^2}\right). \end{cases}$$

*Hint.* For $\mathbb{E}[X] \to \infty$, it might help to split the interval for $p$ into two parts.

(b) Prove that

$$\mathbb{P}(X \geq 1) \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } \mathbb{E}[X] \to 0, \\ 1 & \text{if } \mathbb{E}[X] \to \infty. \end{cases}$$

**Problem 2.6.** Let $r \geq 2$ be given. For any $n \in \mathbb{N}$, $p \in [0,1]$, we denote by $H_r(n,p)$ the random $r$-uniform hypergraph on $n$ vertices, in which each element of $\binom{[n]}{r}$ forms an edge with probability $p$ independently. Let $A$ be the event that each $(r-1)$-tuple is contained in at least one edge. For fixed $\varepsilon > 0$, prove that

$$\mathbb{P}[A] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = (1-\varepsilon)\frac{(r-1)\ln n}{n}, \\ 1 & \text{if } p = (1+\varepsilon)\frac{(r-1)\ln n}{n}. \end{cases}$$