Exercise sheet 3

Exercises for the exercise session on 3 November 2020

Problem 3.1. Let A, B, C be events in a probability space such that

- A and B are independent;
- B and C are independent;
- $A \cap B$ and C are independent.

Prove that A and $B \cap C$ are independent as well. Deduce that any *minimal* dependency graph D for events $A_1, \dots, A_n$ has the following properties.

(i) Every vertex i of D is either isolated (i.e. there are no edges involving i) or has at least one outgoing edge (i.e. an edge (i, j)).

(ii) Either D is edgeless or contains a directed cycle (this cycle might consist of only two edges (i, j) and (j, i)).

Use these properties to show that in the following setup, Lovász Local Lemma *cannot* be applied (regardless of the choice of the dependency graph D). Let Ω be the set of all sequences $s_1, \dots, s_n$ with $s_i \in \{1, \dots, 6\}$ such that $s_1 + \cdots + s_n$ is even. Choose an element from Ω uniformly at random (i.e. we are considering the outcome of throwing a fair die n times, conditioned on the event that the sum of values is even). Denote by A_i the event that s_i is odd.

Problem 3.2. Let $A_1, \dots, A_n$ be events in a probability space and let $D = ([n], E)$ be a directed graph. We say that D is a *negative dependency graph* for $A_1, \dots, A_n$ if the following holds.

> For every $J \subseteq [n]$ and $i \in [n] \setminus J$, if $(i, j) \notin E$ for all $j \in J$ and $\mathbb{P}\left[\bigwedge_{j \in J} \overline{A_j}\right] > 0$, then $\mathbb{P}\left[A_i \,\middle|\, \bigwedge_{j \in J} \overline{A_j}\right] \leq \mathbb{P}(A_i)$.

Replacing 'dependency graph' by 'negative dependency graph' in the statement of the Lovász Local Lemma results in the so-called *Lopsided Lovász Local Lemma.*

(a) Sketch how the proof of the Lovász Local Lemma from the lecture generalises to a proof of the Lopsided Lovász Local Lemma.

(b) Consider the following setup. Let Ω be the set of all permutations of $[n]$. We choose a permutation σ uniformly at random and denote, for each $i \in [n]$, by A_i the event that $\sigma(i) = i$.

Follow the arguments sketched below to prove that the edgeless graph on $[n]$ is a negative dependency graph.

It suffices to show (why?) that

$$\mathbb{P}\left[\bigwedge_{j\in J}\overline{A_j}\,\middle|\,A_i\right] \le \mathbb{P}\left[\bigwedge_{j\in J}\overline{A_j}\right]$$

for all $J \subset [n]$ and $i \in [n] \setminus J$ with $\mathbb{P}[A_i] > 0$. Show that this is equivalent to

$$n\left|A_i \wedge \bigwedge_{j\in J}\overline{A_j}\right| \le \left|\bigwedge_{j\in J}\overline{A_j}\right|.$$

To prove this inequality, for each permutation $\sigma \in A_i \wedge \bigwedge_{j\in J}\overline{A_j}$, define $\sigma_1,\dots,\sigma_n \in \bigwedge_{j\in J}\overline{A_j}$ and prove that $\sigma_k \ne \tau_l$ as soon as $k \ne l$ or $\sigma \ne \tau$.

Problem 3.3. We say that a hypergraph $H = (V, E)$ is *2-colourable* if there exists a colouring of V by two colours so that no edge in E is monochromatic.
Let $k \ge 2$ be given.

(a) Let H be a hypergraph in which every edge has at least k vertices. Suppose that each edge of H intersects at most $d \ge 1$ other edges. Prove that H is 2-colourable if $e(d+1)2^{1-k} \le 1$.

(b) Suppose that H is a hypergraph in which each edge has at least k vertices. For each edge f and each $j \ge k$, denote by $d_{f,j}$ the number of edges of size j that intersect f. Prove that if for each edge f of H

$$8\sum_{j\ge k}\frac{d_{f,j}}{2^j} \le 1,$$

then H is 2-colourable.

Problem 3.4. Let G be a graph and let $d \ge 1$. Suppose that for every vertex v, there exists a list $S(v)$ of precisely $\lceil 2ed \rceil$ 'admissible' colours such that no colour in $S(v)$ is admissible for more than d neighbours of v. Prove that there is a 'proper' colouring of G (i.e. no two adjacent vertices have the same colour) assigning to each vertex an admissible colour.
Hint. The fewer vertices and colours play a role in the probability of a 'bad' event A, the simpler the expression for $\mathbb{P}[A]$ will be.

Problem 3.5. Suppose we place n balls in n bins, where each ball chooses its bin uniformly at random and independently from the other balls.

(a) Prove that for each $\varepsilon > 0$,

$$\mathbb{P}\left[\exists \text{ a bin with at least } \left(\frac{2}{3}+\varepsilon\right)\ln n \text{ balls}\right] = o(1).$$

(b) By how much can we decrease the value $\left(\frac{2}{3}+\varepsilon\right)\ln n$ in (a) so that we can still prove (by the same type of arguments as in (a)) that the probability is $o(1)$?

(c) If we have n^2 balls in total, for what $k = k(n)$ can we prove that

$$\mathbb{P}\left[\exists \text{ a bin with at most } k \text{ balls}\right] = o(1)?$$

Problem 3.6. Define the set $S \subset \mathbb{N}$ by letting each number n be in S with probability $1/2$ independently.

(a) For $k, l \in \mathbb{N}$, we set

$$w_l(k) = \left\lceil \frac{\ln(kl2^{k-1})}{\ln 2} \right\rceil .$$

Denote by A_l the event that there is a $k \geq 2$ such that S contains an arithmetic progression of the form

$$k - b, k, k + b, \ldots, k + (w_l(k) - 2)b.$$

Prove that $\mathbb{P}[A_l] \leq 1/l$ and deduce from this that with probability 1, S does not contain an arithmetic progression of infinite length.

(b) Prove that

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{|S \cap [n]|}{n} = \frac{1}{2}\right] = 1.$$

To that end, for fixed $\varepsilon > 0$ and n, use the Chernoff bounds to find an upper bound for

$$\mathbb{P}\left[\left|\frac{|S \cap [n]|}{n} - \frac{1}{2}\right| \geq \varepsilon\right]$$

and apply a union bound to show that

$$\mathbb{P}\left[\exists n \geq n_0 \text{ with } \left|\frac{|S \cap [n]|}{n} - \frac{1}{2}\right| \geq \varepsilon\right] \overset{n_0 \to \infty}{=} o(1).$$

Where does this strategy fail when we use Chebyshev's inequality instead of Chernoff bounds?

(*Note.* Szemerédi's Theorem states that each $A \subset \mathbb{N}$ with

$$\limsup_{n\to\infty} \frac{|A \cap [n]|}{n} > 0$$

contains infinitely many arithmetic progressions of length k for *every* k. Thus, Problem 3.6 shows that with probability 1, S contains arbitrarily long arithmetic progressions, but no arithmetic progression of infinite length.)