Probabilistic method in combinatorics and algorithmics

WS 2020/21

Exercise sheet 5

Exercises for the exercise session on 26 November 2020

Problem 5.1. A famous result about random graphs states that $G(n,p)$ has a perfect matching with probability $1-o(1)$ whenever n is even and $p \geq \frac{(1+\varepsilon)\ln(n)}{n}$. (Note that this is just large enough in order to guarantee that there are no isolated vertices.) Prove the following weaker result.

Denote by $m(G)$ the size of the largest matching in the graph G and write $\mu := \mathbb{E}[m(G(n,p))]$. Suppose that $p = p(n) \in [0,1]$ is such that $pn \to \infty$. Prove that for every $t > 0$,

$$\mathbb{P}\Big[m\big(G(n,p)\big) \leq \mu - t\sqrt{n-1}\Big] < \exp\left(-\frac{t^2}{2}\right)$$

and show that for every $\varepsilon > 0$,

$$\mu \geq (1-\varepsilon)\frac{pn}{2}$$

if n is large enough.

Problem 5.2. Let $k_0 = k_0(n) \in \mathbb{N}$ be such that

$$\binom{n}{k_0} 2^{-\binom{k_0}{2}} < 1 < \binom{n}{k_0-1} 2^{-\binom{k_0-1}{2}}$$

and let $k := k_0 - 4$. Use Janson's inequality to prove that

$$\mathbb{P}[G(n,1/2) \text{ contains } \textit{every} \text{ graph on } k \text{ vertices as an induced subgraph}] \overset{n\to\infty}{\longrightarrow} 1.$$

Hint. Any ingredients of the applications of Janson's inequality from the lecture can be used without reproving them. Given a graph H on k vertices and a set $S \subset [n]$ of size k, there might be several ways how $G(n,1/2)$ can induce on S a graph isomorphic to H. The calculations will be easier if you consider just one fixed isomorphism.

Problem 5.3. Denote by C_n the number of comparisons that Randomised QuickSort requires to sort an arbitrary, but fixed array $(a[1],\ldots,a[n])$ of distinct numbers. Furthermore, let $k = k(n)$ be a function with $k = \omega(1)$.

(a) Show that
$$\mathbb{P}[C_n \geq kn\ln(n)] = o(1).$$

(b) Use an 'exposure' martingale to show that there exists a constant $c > 0$ with
$$\mathbb{P}[C_n \geq \mathbb{E}[C_n] + k] \leq \exp\left(-\frac{ck^2}{n^3}\right).$$

The following problems are not about the most recent topics of the lecture, but rather spread over the entire course.

Problem 5.4. Recall that $[n] = \{1, \dots, n\}$ is the vertex set of $G(n,p)$. By $d(i,j)$ we denote the distance (that is, the length of the shortest path) between two vertices i, j in $G(n,p)$, where $d(i,j) = \infty$ if they do not lie in the same component of $G(n,p)$. The *diameter* of $G(n,p)$ is defined as

$$\operatorname{diam}(G(n,p)) := \max_{\{i,j\} \in \binom{[n]}{2}} d(i,j).$$

(a) Determine a function $p_0(n)$ so that for any *fixed* pair (i,j),

$$\mathbb{P}[d(i,j) \leq 2] \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = o(p_0), \\ 1 & \text{if } p = \omega(p_0). \end{cases}$$

(b) Show that for any function $h(n)$ with $h = \omega(1)$ but $h = o(\ln(n))$,

$$\mathbb{P}\left[\operatorname{diam}\left(G\left(n, \sqrt{\frac{2\ln(n) + h}{n}}\right)\right) > 2\right] = o(1).$$

Problem 5.5. Let D be a directed graph without loops (i.e. $E(D)$ is a subset of $\{(u,v) \mid u, v \in V(D) \wedge u \neq v\}$) in which each vertex has precisely δ^+ many outgoing edges and at most Δ^- many incoming edges. Suppose that k is a positive integer satisfying

$$e(\delta^+\Delta^- + 1)\left(1 - \frac{1}{k}\right)^{\delta^+} < 1.$$

Prove that there exists a colouring $c\colon V(D) \to \{0, \dots, k-1\}$ such that each vertex $v \in V(D)$ has an outgoing edge (v,w) with $c(w) \equiv c(v) + 1 \bmod k$.
Derive from this that if each vertex of D has *at least* δ^+ outgoing and at most Δ^- incoming edges, then D contains a directed cycle whose length is a multiple of k.

Problem 5.6. Let vectors $v_1, \dots, v_k \in \mathbb{R}^m$ with $\|v_1\| = \dots = \|v_k\| = 1$ be given and define the following random variables.

- Independently for $i = 1, 2, \dots,$ choose $\epsilon_i \in \{1, -1\}$ uniformly at random.
- Recursively for $i = 1, 2, \dots,$ let $d_i \in \{1, \dots, k\}$ be the random variable with

$$\mathbb{P}[d_i = j] := \frac{2^i - 1}{k2^{i-1}} - \sum_{\ell=1}^{i-1} 2^{-\ell \cdot I_{\{d_\ell = j\}}}$$

  for each $j \in \{1, \dots, k\}$.

Prove that for every function $h(n) = \omega(1)$, we have

$$\mathbb{P}\left[\left\|\sum_{i=1}^{n} \epsilon_i v_{d_i}\right\| \geq h(n)\sqrt{n}\right] = o(1).$$